A PROBLEM ON ARC TANGENT RELATIONS*

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1. Notation. It will be convenient to use the letters $a, b, c, d, f, m, n, r, s, t, u, v, w$, with or without suffixes, to denote integers. The letter p will be used to denote prime numbers of the form $4n+1$. For any such p we denote by n_p the smallest n such that $1+n^2$ is a multiple of p. Finally we denote by (x) that value of arc tan x between 0 and $\frac{1}{2}\pi$, except when the context indicates that the brackets have their ordinary significance, for example, in $\Gamma(x+iy)$.

2. Introduction. Integral arc tangents. J. C. P. Miller [1], when engaged in the tabulation of *ln* $\Gamma(x+iy)$, used the relation

$$\ln \Gamma(x + iy + 1) = \ln (x^2 + y^2) + i \arctan (y/x) + \ln \Gamma(x + iy)$$

in order to transfer the argument to a region where an expansion in series converges with sufficient rapidity. He required, therefore, a table of arc tan (a/b) which he constructed and which is being extended by S. Johnston [1], [1a]. He observed identities of the form

$$(3) = 3(1) - (2), \qquad (7) = 2(2) - (1)$$
$$(8) = 2(1) + (3) - (5), \qquad (13) = 5(1) - (2) - (4)$$

and raised the question of the existence of such relations for general (n).

We call (n) irreducible if it cannot be expressed as a finite sum of the form

$$(n) = \sum f_r \times (n_r) \tag{2.1}$$

where the f_r are integers** (positive or negative) and the n_r are (positive) integers less than n. If such a representation exists, as in the cases of (3), (7), (8) and (13) mentioned above, (n) will be called reducible.

The answer to Miller's question is included in the following theorem which has already been announced [12].

THEOREM *A. A condition necessary and sufficient for the reducibility of* (m) *is that all the prime factors of* $1+m^2$ *should occur among the prime factors of* $1+n^2$ *for* $n=1, 2, \cdots, m-1$.

Another condition is given in the next theorem.

THEOREM *B. A condition necessary and sufficient for the reducibility of* (m) *is that the largest prime factor of* $1+m^2$ *should be less than* $2m$.

We shall establish Theorem *A* (in §§6, 7) and deduce Theorem *B* from it

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** It can be deduced from Theorem *A* that nothing new is obtained by considering rational multipliers instead of integral ones.

(in §9). The proof of sufficiency in Theorem A will consist in establishing an algorithm which will completely reduce any reducible integral arc tangent, that is, express it in terms of the (irreducible) arc tangents of certain integers chosen from a base I:

$$1, 2, 4, 5, 6, 9, 10, 11, 12, 14, 15, 16, 19, 20, 22, \cdots .$$

It will be shown in §10 that the number of reducible arc tangents and the number of irreducible arc tangents are both infinite. Numerical evidence (obtained by the use of an unpublished table of factors of $1+n^2$ compiled by J. W. Wrench, Jr.) suggests that if $I(n)$ is the number of terms in I less than n then the ratio $I(n)/n$ has a limit about 0.71. This problem has been discussed elsewhere [13].

Two short tables of reductions are given. Their contents and construction are described in §§11, 12.

3. Rational arc tangents. D. H. Lehmer [2] has shown how to express arc cot (a/b) as a finite sum of the form*

$$\text{arc cot } (a/b) = \text{arc cot } n_0 - \text{arc cot } n_1 + \text{arc cot } n_2 - \cdots \tag{3.1}$$

where the integers n_i are obtained by an arc cotangent algorithm. This algorithm consists in the recurrence formulae

$$a_i = n_i b_i + b_{i+1} \ (0 \leqq b_{i+1} < b_i), \qquad a_{i+1} = a_i n_i + b_i$$

with the initial conditions $a_0 = a$, $b_0 = b$.

It follows that any (a/b) can be expressed as a finite sum of the form (2.1). The set I therefore constitutes a base for the rational arc tangents as well as the integral arc tangents. The problem of the existence of such a base has been studied by Gauss ([3], p. 523; [5], p. 74). He was apparently aware that any (a/b) could be expressed in terms of the "primitive arcs," that is, the arc tangents of certain rational numbers which constitute a base R:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{4}{1}, \frac{5}{2}, \frac{6}{1}, \frac{5}{4}, \cdots .$$

The connection between the "rational base" R and the "integral base" I is indicated by the following remark: Corresponding to any p we have in I a term n_p and in R a term (a/b) where a, b are determined uniquely by the relations $a > b > 0$, $a^2 + b^2 = p$ and we can show that (a/b) can be expressed in terms of

* There is much to be said for using the arc cot as the basis for such considerations as those of the present paper. For instance, to have arc cot $n \to 0$ as $n \to \infty$ is often more convenient than arc tan $n \to \frac{1}{2}\pi$. Adoption of the arc cot might have prevented an error in the coefficient of (1) in the reduction of (1, 40333, 78718) which was fortunately detected by Professor Lehmer in the proofs of the announcement [12]. Nevertheless it has been decided to retain the original arc tan basis as with it checking is slightly easier. However extra columns have been added to the tables to give the representations of arc cot n in terms of arc cot n_i: in these the coefficients of (1) are somewhat smaller.

(n_p) and (m) with $m<n_p$.

4. The cases $m=2$ and $m=3$. The idea behind the proof of Theorem A has been used by many authors ([2], [3], [4], [5], [6], [7]). It is made clear in the following discussion of the cases $m=2$, $m=3$. The problem and the results do not appear to have been explicitly stated in the present form before although Størmer [5] has considered the solution of such equations as

$$f \times (1) = \sum_{r=1}^{s} f_r \times (m_r)$$

where s and f are fixed; his Theorem 2 ([5], p. 20) is essentially the same as Theorem A. He has given extensive tables of solutions. A similar problem has been studied by Bennett [11].

If (2) is reducible we must have $(2)=r(1)$ for some integer r. In this case the complex numbers $1+2i$ and $(1+i)^r$, having the same argument, must have a real ratio which is clearly a rational number, which we will denote by m/n. Taking moduli in the equation $(1+2i)=(m/n)(1+i)^r$ we have

$$5n^2 = 2^r m^2$$

which is manifestly impossible. Hence (2) is irreducible. If (3) is reducible we must have $(3)=r(1)+s(2)$ for some integers r, s. As before, we find

$$10n^2 = 2^r 5^s m^2.$$

This is soluble in integers. We find that the relevant solution is $r=3$, $s=1$, which gives

$$(3) = 3(1) - (2).$$

5. Auxiliary material. In the discussion of the general case we require certain results about complex or Gaussian integers, that is, complex numbers of the form $a+ib$, where a and b are integers. The results required will be found in standard textbooks (*e.g.*, [8] ch. 12, 14, 15). One complex integer is said to be a factor of another if the quotient, a complex number, is a complex integer. A complex integer e whose reciprocal e^{-1} is also a complex integer is called a unit; if e is a unit, any complex integer $a+ib$ and the complex integer $e(a+ib)$ are said to be associates. The only units among the complex integers are the numbers ± 1, $\pm i$. A complex integer is said to be prime if it is not 0, not a unit, and is divisible only by numbers associated with itself or with 1. Any complex integer can be expressed (uniquely apart from the order of the factors, the presence of units and the ambiguities between associated primes) as a product of primes. This decomposition can be carried out in a finite number of steps. A prime complex integer is either $1+i$ or one of its associates, an ordinary prime number of the form $4n+3$ or one of its associates, factors $a+ib$ of an ordinary prime number $4n+1$. It is known that the factors in the latter case are essentially unique; if one is $a+ib$ the others must be units or associated with $a-ib$.

We shall require in particular the following lemmas.

LEMMA 1. (Cf. [5], p. 8.) *If u^2+v^2 is a prime factor of $1+n^2$ then $u+iv$ is a factor of $1\pm in$ according as u^2+v^2 is a factor of $nu\mp v$.*

Proof. In the case $u^2+v^2=2$ it is easy to see that $u+iv$ divides both $1\pm in$. We shall not require this case in our proof and shall assume that $u^2+v^2>2$. We have

$$\frac{1+in}{u+iv}=\frac{(u\pm nv)+i(-v\pm nu)}{u^2+v^2}.$$

Now $\mp n(u\pm nv)=-v(1+n^2)-(-v\pm nu)$; so u^2+v^2 is a factor of $\mp n(u\pm nv)$ if it is a factor of $-v\pm nu$. Assume u^2+v^2 is a factor of $-v\pm nu$. Then u^2+v^2 either divides n or $u\pm nv$. The first alternative is impossible for if u^2+v^2 was a factor of n it would also be a factor of n^2 which is impossible since it is a factor of $1+n^2$. It remains only to show that u^2+v^2 is a factor of $-v\pm nu$. It will be enough to show that u^2+v^2 divides the product $(-v+nu)(-v-nu)=v^2-n^2u^2=u^2+v^2-u^2(1+n^2)$, and this is obvious.

LEMMA 2. *Corresponding to any p there is an integer $m<p$ such that for an $x=n_p\leqq\frac{1}{2}(p-1)$ we have*

$$1+x^2=mp.$$

This is well-known (e.g. [8], p. 70).

We notice that there is an infinite number of primes of the form $4n+1$ and that every odd prime factor of a^2+b^2 (when a, b have no common factor) is of this form ([8], p. 13 and p. 297).

6. Proof of necessity. Suppose $(m)=\sum_{r=1}^{s}\pm(m_r)$ where the m_r, not necessarily all different, are all less than m. This means that

$$\arg(1+im)=\arg\prod_{r=1}^{s}(1\pm im_r)$$

and so the ratio

$$\frac{\prod_{r=1}^{s}(1-im_r)}{1+im}$$

is a real rational number. Since the numerator is of the form $n'+in''$ and the denominator of the form $1+im$ it is clear that the quotient is the integer n'. Taking moduli we find that

$$n'^2(1+m^2)=\prod_{r=1}^{s}(1+m_r^2)$$

from which the necessity of the condition stated follows.

EXAMPLES. In the case $m=7$, since $(7)=2(2)-(1)$ we have

$$n' = (1+2i)^2(1-i)/(1+7i) = 1.$$

In the case $m=286$, since $(286)=4(1)+(23)-(28)-(235)$ we have

$$n' = (1+i)^4(1+23i)(1-28i)(1-235i)/(1+286i) = 2^3\cdot 5\cdot 53.$$

In the case $m=313$, since $(313)=(1)-2(2)+(10)+(22)$ we have

$$n' = (1+i)(1-2i)^2(1+10i)(1+22i)/(1+313i) = 5.$$

7. Proof of sufficiency. It will be enough to express $1+im$ as the product of a real number and complex integers of the form $1+iw_r$ $(0<w_r<m)$, being given that each prime factor p of $1+m^2$ is a factor of $1+m_p^2$ for some $m_p<m$.

We begin by expressing $1+im$ as a product of prime complex integers. Take the prime factor $u+iv$ of largest modulus* which is neither of the required form nor associated with a number of the required form; suppose it occurs to the power $f\geqq 1$. We shall show how to replace this by a factor of the required form and one of smaller modulus. Suppose we have

$$1 + im = (u+iv)^f(c+id). \tag{7.1}$$

Consider the following equation for a and b, where w is yet to be defined:

$$(u+iv)(a+ib) = 1+iw. \tag{7.2}$$

This gives

$$au - bv = 1, \qquad av + bu = w. \tag{7.3}$$

We now observe that it is possible to choose solutions a and b of (7.3) such that $|w|<m$. Since u^2+v^2 is a prime factor of $1+m^2$ it is therefore also a factor of $1+n^2$ for some $n<m$. This implies, by virtue of our lemma, that $u+iv$ is a factor of $1\pm in$. We can certainly determine a, b such that (7.2) and (7.3) are satisfied with $w=\pm n$ and so with $|w|=n<m$.

We now proceed as follows. Multiply (7.1) across by $(a^2+b^2)^f$ to get

$$(1+im)(a^2+b^2)^f = (1+iw)^f(a-ib)^f(c+id). \tag{7.4}$$

Now take the prime factor u_1+iv_1 of largest modulus of $(a-ib)^f(c+id)$. Suppose it occurs to the power $f_1\geqq 1$ and that we have

$$(a-ib)^f(c+id) = (u_1+iv_1)^{f_1}(c_1+id_1).$$

Consider the equation for a_1, b_1

$$(u_1+iv_1)(a_1+ib_1) = 1+iw_1. \tag{7.5}$$

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* There is only one factor of largest modulus (apart from associates) for if $u+iv$ and $u-iv$ were factors we would have u^2+v^2 a factor of 1.

We observe that it is possible to choose integers a_1, b_1 to satisfy (7.5) with a w_1 such that $|w_1| < m$. Since u_1+iv_1 is a factor either of $1+im$ or of $a-ib$ it follows that $u_1^2+v_1^2$ is a factor of $1+m^2$ or of a^2+b^2. In the first case the argument already given in the discussion of w applies; in the second, because of (7.2) a^2+b^2 is a factor of $1+w^2$ and since $|w| < m$ a similar argument applies.

Multiplying (7.4) across by $(a_1^2+b_1^2)^{f_1}$ we find

$$(1 + im)(a^2 + b^2)^f(a_1^2 + b_1^2)^{f_1} = (1 + iw)^f(1 + iw_1)^{f_1}(a_1 - ib_1)^{f_1}(c_1 + id_1).$$

Continuing in this way it is clear that we come, after a finite number of steps, to a relation

$$(1 + im)\prod (a_i^2 + b_i^2) = \prod (1 + iw_r)^{f_r}$$

from which the actual reduction for (m) follows by equating the argument of the two sides; hence

$$(m) = \sum f_r \times (w_r).$$

It may be necessary to adjust* this by addition of an appropriate multiple of 4(1) because of our convention that $0 < (x) < \frac{1}{2}\pi$. This completes the proof of Theorem A.

8. Examples of reductions. Two examples will make clear how this reduction proceeds or comes to an end.

EXAMPLE 1. Since $1+286i = (11+6i)\ (11+20i)$ we consider the equation $(11+20i)\ (a+ib) = 1+iw$. The general solution of this is

$$a = 11 - 20n, \qquad b = 6 - 11n, \qquad w = 286 - 521n.$$

We choose $n=1$, $w=-235$, and continue. Since

$$(1 + 286i)(106) = (1 - 235i)(-9 + 5i)(11 + 6i)$$

we consider the equation $(11+6i)(a+ib) = 1+iw$. The general solution of this is

$$a = -1 + 11n, \qquad b = -2 + 6n, \qquad w = -28 + 132n.$$

We choose $n=0$, $w=-28$, and continue. Since

$$(1 + 286i)(106)(5) = (1 - 235i)(1 - 28i)(-1 + 2i)(-9 + 5i)$$

we consider the equation $(-9+5i)(a+ib) = 1+iw$. The general solution of this is

* The adjustment can be carried out without reference to tables by carefully examining each stage of the reduction process. This is discussed in detail elsewhere [14].

$$a = 1 - 5n, \qquad b = -2 - 9n, \qquad w = 23 - 106n.$$

We choose $n=0$ and have now obtained

$$(1 + 286i)(106)(5)(5) = (1 - 235i)(1 - 28i)(1 + 23i)(-1 + 2i)(1 + 2i)$$

whence, apart from multiples of $\pi = 4(1)$,

$$(286) = (23) - (28) - (235).$$

Checking numerically from tables we see that the required reduction is

$$(286) = 4(1) + (23) - (28) - (235).$$

EXAMPLE 2. Since $1+23i=(1+3i)(7+2i)$ we consider the equation $(7+2i)(a+ib)=1+wi$. The general solution of this is $a=1-2n$, $b=3-7n$, $w=23-53n$. It is clear we cannot get $|w|<23$ by choice of n and so (23) is irreducible.

9. Deduction of Theorem *B* from Theorem *A*. Suppose that every prime factor of $1+m^2$ is less than $2m$. Then, by Lemma 2, corresponding to each prime factor p of $1+m^2$, we have $n_p \leqq \frac{1}{2}(2m-1) < m$. That is, each such p occurs as a factor of $1+n^2$ for some $n=1, 2, \cdots, m-1$. Theorem *A* now shows that (m) is reducible.

On the other hand suppose that (m) is irreducible. We shall show that each prime factor p of $1+m^2$ is less than $2m$. For suppose we had $p \geqq 2m$. By Theorem *A* there is an $n_p < m$ such that $1+n_p^2$ is a multiple of p. Since $1+m^2$ is also a multiple of p, so is

$$m^2 - n_p^2 = (m - n_p)(m + n_p).$$

Now since one of these factors is less than m, the other less than $2m$, and neither vanish, we have arrived at a contradiction. This completes the proof of Theorem *B*.

10. The number of reducible and irreducible arc tangents. We shall now show that the number of irreducible arc tangents is infinite. If the number is finite choose a p greater than all the prime factors of the corresponding set of numbers $1+m^2$. In virtue of Lemma 2 we can choose an n such that $1+n^2$ contains p as a factor. It follows that (n) cannot be expressed in terms of the finite "base"; (n), therefore, is irreducible or has an irreducible component not in the given base.

We next show that the number of reducible arc tangents is infinite. In fact, corresponding to a given p we shall show that $(p-n_p)$ is reducible. For, by Lemma 2,

$$p - n_p < n_p < m = \frac{1 + n_p^2}{p}$$

and

$$(p - n_p) + (n_p) = \left(\frac{p - n_p + n_p}{1 - pn_p + n_p^2}\right) = \left(\frac{1}{m - n_p}\right) = 2(1) - (m - n_p)$$

so that $(p-n_p)=2(1)+(n_p-m)-(n_p)$. We complete the proof with the remark that we can certainly choose an infinite subsequence of the p for which the $p-n_p$ are all different.

EXAMPLES. If $p=13$ then $n_p=5$ and $m=2$; we find $(8)=2(1)+(3)-(5)$ $=5(1)-(2)-(5)$.

If $p=37$ then $n_p=6$ and $m=1$; we find $(31)=2(1)+(5)-(6)$.

11. Description and construction of Table I. In Table I is set out the complete reductions of all reducible arc tangents (n), with $n \leqq 342$. The factor table used in its construction was that issued by the British Association for the Advancement of Science (Cambridge, 1935). This table covers the range up to 100,000.

There is no difficulty in extending this table further by use of the algorithm described or by the following process, which is often more convenient.

We suppose that the factorization of $1+m^2$ is known and that we have a table of n_p against p [15]. If a prime factor of $1+m^2$ is greater than $2m$ then (m) is irreducible. If not, we start from the highest factor p and build up $1+m^2$ into a product of numbers $1+n^2$ with $n<m$. For instance, $1+286^2=81797=157\cdot 521$. Since $521<2.286=572$, (286) is reducible. We note that $2\cdot 53\cdot 521=1+235^2$ and that $5\cdot 157=1+28^2$.

We now multiply across by $(2\cdot 53)^2$ to get

$$(1+286^2)(2\cdot 53)^2=2\cdot 53\cdot 157(1+235^2).$$

We next observe that $2\cdot 5\cdot 53=1+23^2$, and so

$$(1+286^2)(2\cdot 53)^2\cdot 5^2=2\cdot 5\cdot 53(1+28^2)(1+235^2)=(1+23^2)(1+28^2)(1+235^2).$$

We now know that (23), (28), (235) are the components of (286) and the signs can be determined by congruence relations similar to those given in Lemma 1. It remains to decide whether or not a multiple of $\pi=4(1)$ is to be added. This can be decided without use of tables [cf. 14].

In addition to the complete reduction of (n) for each n there is in Table I a column giving a coefficient c_n which enables the expression of arc cot n in terms of irreducible arc cotangents to be written down. In fact, if

$$(n)=\sum f_r\times(n_r)+f\times(1),$$

then

$$\text{arc cot } n = \sum f_r \text{ arc cot } n_r + c_n \text{ arc cot } 1$$

where

$$c_n = 2 - f - 2\sum f_r.$$

TABLE I

n	c_n	(n)	n	c_n	(n)
3	1	$3(1)-(2)$	182	−3	$-7(1)+5(2)+(23)$
7	−1	$-(1)+2(2)$	183	0	$4(1)+(3)-(4)-(14)$
8	1	$5(1)-(2)-(5)$	185	0	$2(1)+(28)-(33)$
13	1	$5(1)-(2)-(4)$	187	−2	$-2(1)+3(2)+(5)-(82)$
17	−1	$(1)+2(2)-(12)$	189	1	$7(1)-2(2)+(5)-(23)-(148)$
18	1	$3(1)-2(2)+(5)$	191	−1	$-(1)+(2)+(4)+(6)-(12)$
21	0	$2(1)+(4)-(5)$	192	−1	$3(1)+2(2)-(10)-(27)$
30	1	$7(1)-(2)-(4)-(23)$	193	2	$6(1)-3(2)-(5)+(44)$
31	0	$2(1)+(5)-(6)$	200	0	$-(4)+(5)+(19)$
32	−1	$(1)+2(2)-(9)$	203	1	$9(1)-(2)-(5)-(9)-(114)$
38	0	$-(2)+2(4)$	211	0	$2(1)+(14)-(15)$
41	1	$(1)-2(2)+2(12)$	212	−1	$-5(1)+(2)+(5)+(10)+(34)$
43	1	$3(1)-2(2)+(6)$	213	1	$7(1)-2(2)+(5)-(23)-(136)$
46	−1	$3(1)+2(2)-(12)-(27)$	216	1	$5(1)-(2)-(5)-(6)+(22)$
47	0	$4(1)+(2)-(4)-(5)$	217	0	$6(1)+(2)-(4)-(5)-(60)$
50	0	$2(1)+(9)-(11)$	228	1	$5(1)-2(2)+(6)-(53)$
55	0	$4(1)+(4)-(5)-(34)$	233	1	$9(1)-(2)-(5)-(11)-(34)$
57	−2	$-4(1)+3(2)+(5)$	237	−1	$3(1)+2(2)-(9)-(37)$
68	2	$8(1)-3(2)-(6)$	239	−1	$-5(1)+4(5)$
70	0	$-2(1)-(2)+2(5)+(12)$	241	0	$2(1)+(15)-(16)$
72	−1	$-3(1)+(2)+(4)+(11)$	242	0	$4(1)+(4)-(5)-(23)$
73	1	$7(1)-(2)-(5)-(9)$	251	0	$2(1)+(2)-2(4)+(33)$
75	−1	$3(1)+2(2)-(12)-(22)$	253	1	$5(1)-(2)-2(6)+(80)$
76	0	$2(1)+(23)-(33)$	254	0	$6(1)+(5)-(6)-(44)-(179)$
83	1	$5(1)-2(2)+(5)-(23)$	255	0	$-(5)+(9)+(11)$
91	0	$2(1)+(9)-(10)$	265	−1	$-(1)+(2)+(5)+(6)-(27)$
93	1	$5(1)-2(2)+(6)-(80)$	266	1	$7(1)-2(2)+(6)-(80)-(143)$
98	1	$7(1)-(2)-(4)-(15)$	268	2	$10(1)-2(2)-(4)-2(5)$
99	1	$5(1)-(2)-2(5)+(12)$	269	−1	$5(1)+2(2)-(12)-(22)-(104)$
100	0	$2(1)+(27)-(37)$	273	1	$(1)-2(2)+(12)+(16)$
105	0	$4(1)+(5)-(6)-(44)$	274	0	$2(1)+(81)-(115)$
111	0	$2(1)+(10)-(11)$	278	0	$4(1)+(5)-(9)-(12)$
112	0	$6(1)+(2)-(4)-(5)-(81)$	286	0	$4(1)+(23)-(28)-(235)$
117	−1	$-3(1)+(2)+2(6)$	288	0	$2(1)+(23)-(25)$
119	0	$2(1)+(22)-(27)$	293	1	$5(1)-2(2)+(4)-(10)$
122	−2	$-2(1)+3(2)+(5)-(107)$	294	1	$9(1)-(2)-(5)-(11)-(33)$
123	0	$-2(1)-(2)+(4)+(5)+(34)$	301	0	$-(2)+(4)+(5)+(34)-(208)$
128	1	$(1)-2(2)+(12)+(15)$	302	0	$-(4)+(6)+(12)$
129	0	$2(1)+(23)-(28)$	305	0	$2(1)+(64)-(81)$
132	−1	$-(1)+2(2)-(4)+(9)$	307	2	$4(2)-(5)-(12)$
133	0	$2(1)+(11)-(12)$	311	0	$2(1)+(37)-(42)$
142	−1	$-(1)+2(2)-(6)+(33)$	313	1	$(1)-2(2)+(10)+(22)$
144	0	$6(1)+(4)-(5)-(34)-(89)$	319	1	$5(1)-(2)-(4)-(9)+(27)$
155	1	$9(1)-(2)-(5)-(9)-(138)$	322	0	$4(1)+(2)-(4)-(5)-(34)+(89)$
157	−1	$-3(1)+(2)+(4)+(12)$	327	−1	$-(1)+(2)+2(4)-(6)$
162	−1	$3(1)+2(2)-(12)-(19)$	334	0	$2(1)+(53)-(63)$
172	−1	$3(1)+2(2)-(11)-(22)$	336	1	$5(1)-(2)-(4)-(12)+(107)$
173	1	$(1)-2(2)+(9)+(27)$	337	−1	$-5(1)+(2)+(5)+(9)+(60)$
174	0	$-2(1)-(2)+(4)+(5)+(37)$	338	0	$2(1)+(25)-(27)$
177	0	$6(1)+(2)-(4)-(5)-(64)$	342	−1	$-3(1)+2(2)-(5)+(28)+(44)$

TABLE II

p	n_p	m	a	b	c_n	(a/b)
2	1	1	1	1	1	(1)
5	2	1	2	1	0	(2)
13	5	2	3	2	1	3(1)−(5)
17	4	1	4	1	0	(4)
29	12	5	5	2	0	2(1)+(2)−(12)
37	6	1	6	1	0	(6)
41	9	2	5	4	1	3(1)−(9)
53	23	2.5	7	2	1	5(1)−(2)−(23)
61	11	2	6	5	1	3(1)−(11)
73	27	2.5	8	3	1	(1)−(2)+(27)
89	34	13	8	5	1	5(1)−(5)−(34)
97	22	5	9	4	0	2(1)+(2)−(22)
101	10	1	10	1	0	(10)
109	33	2.5	10	3	0	2(1)+(3)−(33)
113	15	2	8	7	1	3(1)−(15)
137	37	2.5	11	4	1	(1)−(2)+(37)
149	44	13	10	7	1	(1)−(5)+(44)
157	28	5	11	6	0	−2(1)+(2)+(28)
173	80	37	13	2	0	2(1)+(6)−(80)
181	19	2	10	9	1	3(1)−(19)
193	81	2.17	12	7	1	5(1)−(4)−(81)
197	14	1	14	1	0	(14)
229	107	2.5^2	15	2	−1	(1)+2(2)−(107)
233	89	2.17	13	8	1	(1)−(4)+(89)
241	64	17	15	4	0	−2(1)+(4)+(64)
257	16	1	16	1	0	(16)
269	82	5^2	13	10	2	2(1)−2(2)+(82)
277	60	13	14	9	1	5(1)−(5)−(60)
281	53	2.5	16	5	1	5(1)−(2)−(53)
293	138	5.13	17	2	1	7(1)−(2)−(5)−(138)
313	25	2	13	12	1	3(1)−(25)
317	114	41	14	11	1	5(1)−(9)−(114)
337	148	5.13	16	9	1	3(1)−(2)+(5)−(148)
349	136	53	18	5	1	7(1)−(2)−(23)−(136)
353	42	5	17	8	0	2(1)+(2)−(42)
373	104	29	18	7	0	4(1)+(2)−(12)−(104)
389	115	2.17	17	10	1	5(1)−(4)−(115)
397	63	2.5	19	6	1	5(1)−(2)−(63)
401	20	1	20	1	0	(20)
409	143	2.5^2	20	3	−1	−3(1)+2(2)+(143)

TABLE II (Continued)

p	n_p	m	a	b	c_n	(a/b)
421	29	2	15	14	1	$3(1)-(29)$
433	179	2.37	17	12	1	$5(1)-(6)-(179)$
449	67	2.5	20	7	1	$(1)-(2)+(67)$
457	109	2.13	21	4	0	$2(1)+(5)-(109)$
461	48	5	19	10	1	$(1)-(3)+(48)$

12. Description and construction of Table II. In Table II is set out the complete reductions of (a/b) for all a, b without common factor and satisfying $a>b>0$, $a^2+b^2<500$. The table contains, in addition, for each $a^2+b^2=p\leqq 461$, the corresponding n_p and the residual factor $m=(1+n_p^2)/p$. It also gives the coefficient c_p by means of which the reduction of arc tan (a/b) can be transformed into a reduction of arc cot (a/b). If

$$(a/b) = \sum f_r \times (n_r) + f \times (1)$$

then

$$\text{arc cot } (a/b) = \sum f_r \text{ arc cot } n_r + c_p \text{ arc cot } 1.$$

This table was constructed by the following process. If a, b are integers without common factor and such that $a>b>0$ we find* non-negative integers $a_1<a$, $b_1<b$ such that

$$ab_1 - ba_1 = \pm 1.$$

We then observe that, where the ambiguous sign is the same as in the last equation,

$$\frac{(a/b) \pm (a_1a + b_1b)}{1 \mp (a/b)(a_1a + b_1b)} = \frac{\pm a(ab_1 - ba_1) \pm b(a_1a + b_1b)}{\pm b(ab_1 - ba_1) \mp a(a_1a + b_1b)} = \frac{-b_1}{a_1}.$$

Hence we have

$$(a/b) = -2(1) + (a_1/b_1) \mp (a_1a + b_1b),$$

expressing (a/b) in terms of arc tangents of integers and the arc tangent of a_1/b_1, a fraction with smaller denominator than a/b. Successive applications of this process will lead to an expression of (a/b) as a sum of arc tangents of integers; these can be expressed in terms of the irreducible arc tangents by the methods already described.

* The forthcoming volume in the series issued by the Royal Society, Mathematical Tables Committee, Farey Series of Order 1025 by E. H. Neville, will enable solutions of such equations to be obtained readily.

EXAMPLE. If $a=100$, $b=17$ then $a_1=47$, $b_1=8$. Since $8\cdot 100-17\cdot 47=1$ we have

$$(100/17) = -(4836) - (8/47) = -(4836) + 2(1) + (47/8).$$

Next if $a=47$, $b=8$ then $a_1=6$, $b_1=1$. Since $1\cdot 47-6\cdot 8=-1$, we have

$$(47/8) = (290) - (1/6) = (290) - 2(1) + (6).$$

Hence, referring to tables or otherwise determining [14] the appropriate multiple of $\pi=4(1)$ to add, we find $(100/17)=(6)+(290)-(4836)$. Each of these components can be proved irreducible.

This is a much more convenient process than obtaining the Lehmer-reduction and then completely reducing it because of the rapidity at which the numbers in Lehmer's algorithm increase. To see this we write down the decomposition (3.1) corresponding to the case just discussed:

$$\text{arc cot } (100/17) = \text{arc cot } 5 - \text{arc cot } 34 + \text{arc cot } 2513 - \text{arc cot } 22105608.$$

13. Acknowledgement. The author is indebted to E. M. Wilson of the Royal Naval Scientific Service for help in checking the reductions. This was done by combining the components according to the addition formula for the arc tangent and also numerically, using the standard tables [9], [10]. He is also indebted to J. C. P. Miller for reading an early version of the paper.

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AN APPROXIMATION TO THE QUOTIENT OF GAMMA FUNCTIONS

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1. A modification of Wallis' formula. A commonly used approximation for the middle term of the expansion of $(\frac{1}{2}+\frac{1}{2})^{2n}$ is

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\sim(n\pi)^{-1/2}. \tag{1}$$

This is equivalent to the product formula of Wallis

$$\frac{\pi}{2}=\lim_{n=\infty}\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n-2}{2n-1}\cdot\frac{2n}{2n-1}\right), \tag{2}$$

or to the approximation

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\sim n^{1/2}. \tag{3}$$

A much closer approximation, with a relative error less than $\frac{1}{2}(4n+1)^{-4}$ is obtained by replacing $n^{1/2}$ in (1) and (3) by $(n^2+\frac{1}{2}n+\frac{1}{8})^{1/4}$. We then have

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\sim\left(n^2+\frac{n}{2}+\frac{1}{8}\right)^{1/4}. \tag{4}$$

Upon squaring both members of (4), we obtain for integral n the following approximation for π, which is a modification of Wallis' formula:

$$\pi_n=\left(\frac{2}{1}\cdot\frac{4}{3}\cdots\frac{2n}{2n-1}\right)^2\left(n^2+\frac{n}{2}+\frac{1}{8}\right)^{-1/2};\qquad \lim_{n=\infty}\pi_n=\pi. \tag{5}$$

This approximation π_n can be shown to satisfy the inequality

$$\pi-\frac{\pi}{(4n+1)^4}<\pi_n<\pi-\frac{0.74\pi}{(4n+1)^4},\qquad \text{for } n\geqq 1. \tag{6}$$

2. The Gamma function quotient. Formula (4) is but a special case of the following excellent approximation to the quotient of two nearby values of the Gamma function:

$$\frac{\Gamma\left(n+\frac{1+u}{2}\right)}{\Gamma\left(n+\frac{1-u}{2}\right)}\sim\left(n^2+\frac{1-u^2}{12}\right)^{u/2}. \tag{7}$$

Formula (7) is an exact equality for all $n\geqq 1$, when $u=0,\ \pm 1,\ \pm 2$. Its logarithmic relative error $E_n(u)$, and a related correction factor $F_n(u)$ may be de-

fined by the equations

$$\frac{\Gamma\left(n+\frac{1+u}{2}\right)}{\Gamma\left(n+\frac{1-u}{2}\right)}=\left(n^2+\frac{1+u^2}{12}\right)^{u/2}e^{-E_n(u)}, \tag{8}$$

$$E_n(u)=\frac{u(1-u^2)(4-u^2)}{6!n^4}F_n(u). \tag{9}$$

We shall show that $F_n(u)$ is an even function of both n and u, and that it is positive but less than 1 for $n\geqq 1$, $|u|\leqq 1$. A good estimate for $F_n(u)$ will be given in (39) which is sufficiently accurate to define $E_n(u)$ to six decimals, and is a simple rational function of n^2 and u^2. To give the reader an idea of the magnitude of $F_n(u)$ we now tabulate in table (10) certain of its values, the computation of which will be discussed in Section 4.

Table for $F_n(u)$

(10)

u \ n	1	2	3	4	∞
0	0.636	0.865	0.933	0.960	1.000
0.5	0.658	0.875	0.937	0.963	1.000
1.0	0.734	0.904	0.952	0.972	1.000
2.0	1.257	1.040	1.015	1.008	1.000

From (8), (9), and (10) it is evident that the approximation (7) is good to at least four significant figures for $n>2$, $|u|\leqq 1$.

3. The asymptotic expansion. To obtain the approximation (7) and estimate the error factors $E_n(u)$ and $F_n(u)$ in (8) and (9), we start with the asymptotic expansion in powers of $1/n$ of a related function $G(x, n)$ and the power sum polynomials $S_\nu(x)$ which it generates. Let

$$G(x, n)=\ln\frac{\Gamma(n+x)}{n^x\Gamma(n)}=-\sum_{\nu=1}^{\infty}\frac{S_\nu(x)}{\nu(-n)^\nu}. \tag{11}$$

For integral values of x, and for $n>|x|$, we may expand $G(x, n)$ in a convergent power series in $1/n$, thus:

$$G(x, n)=\sum_{k=1}^{x-1}\ln\left(1+\frac{k}{n}\right)=-\sum_{\nu=1}^{\infty}\left(\frac{1}{\nu(-n)^\nu}\sum_{k=1}^{x-1}k^\nu\right). \tag{12}$$

Comparing (11) and (12) we see that $S_\nu(x)$ are the power sum polynomials:

$$S_\nu(x) = \sum_{k=1}^{x-1} k^\nu = \frac{x^{\nu+1}}{\nu+1} - \frac{1}{2}x^\nu + \sum_{r=1}^{[\nu/2]} \frac{(-1)^{r-1}B_r}{2r}\binom{\nu}{2r-1} x^{\nu+1-2r} \tag{13}$$

where the coefficients B_r are the Bernoulli numbers

$$B_1 = 1/6,\ B_2 = 1/30,\ B_3 = 1/42,\ B_4 = 1/30,\ B_5 = 5/66, \text{ etc.} \tag{14}$$

If x is not an integer, the formulas (11) and (13) are still valid for $n > |x|$, but the expansion (12) which was used in the proof of (13) must be replaced by the somewhat more complicated asymptotic expansion (15), which is called Stirling's formula with the remainder, namely,

$$\begin{aligned} &\ln \Gamma(n) = \tfrac{1}{2}\ln(2\pi) + (n - \tfrac{1}{2})\ln n - n + R(n), \quad \text{where} \\ &R(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n^{2k-1}} \frac{B_k}{2k(2k-1)}. \end{aligned} \tag{15}$$

Thus,

$$\begin{aligned} G(x, n) &= (n + x - \tfrac{1}{2})\ln(n + x) - (n - \tfrac{1}{2})\ln n + R(n + x) - R(n) \\ &= -\sum_{\nu=1}^{\infty} S_\nu(x)/\nu(-n)^\nu. \end{aligned} \tag{16}$$

Taking the logarithm of the left member of (7) and subtracting $u \ln n$ we obtain the function

$$\begin{aligned} &G\left(\frac{1+u}{2}, n\right) - G\left(\frac{1-u}{2}, n\right) \\ &\qquad = -\sum_{\nu=1}^{\infty} \left\{ S_\nu\left(\frac{1+u}{2}\right) - S_\nu\left(\frac{1-u}{2}\right) \right\} / \nu(-n)^\nu. \end{aligned} \tag{17}$$

This is obviously an odd function of u. But some other properties of this function, such as the fact that it is an even function of n, are most easily derived by using an exponential generating function $S(x, t)$ for the polynomials $S_\nu(x)$, namely,

$$S(x, t) = \frac{e^{xt} - 1}{e^t - 1} = \sum_{\nu=0}^{\infty} S_\nu(x) \frac{t^\nu}{\nu!}. \tag{18}$$

It may interest some readers to mention that the function $-G(x, -n)$ is the Laplace transform of the function $[S(x, t) - x]/t$. Certain difficulties with convergence are avoided by working with the exponential series (18) instead of the logarithmic series (16) to derive the properties of the power sum polynomials $S_\nu(x)$. Thus if we expand the left members of the identity

$$S(1 - x, -t) + S(x, t) \equiv 1, \tag{19}$$

using (18), compare coefficients, and set $x=(1+u)/2$, we obtain

$$(-1)^{\nu}S_{\nu}\left(\frac{1-u}{2}\right)+S_{\nu}\left(\frac{1+u}{2}\right)=0, \qquad \text{for } \nu>0. \tag{20}$$

Hence the odd powers of $1/n$ drop out of the expansion (17). The polynomial coefficients for even powers of $1/n$ may be generated by using another identity, namely,

$$\frac{1}{u}\left\{S\left(\frac{1+u}{2},\ t\right)-S\left(\frac{1-u}{2},\ t\right)\right\} = \frac{\sinh \frac{1}{2}ut}{u\sinh \frac{1}{2}t}=1+\frac{u^2-1}{12}\frac{t^2}{2!}+\frac{(u^2-1)(3u^2-7)}{2\cdot 5!}\frac{t^4}{4!}+\cdots. \tag{21}$$

Observing the first two terms in the expansion just given, and observing that the function (21) has the value 1 for $u=1$ and the value $\cosh \frac{1}{2}t$ for $u=2$, we may preserve these properties by using the approximating function $\cosh wt$, where $w^2=(u^2-1)/12$. The remainder after subtracting $\cosh wt$ from (21) is a function $P(u, t)$ which is small for small t and vanishes for all t when $u=\pm 1$ or ± 2. It generates certain polynomials $P_{\nu}(u)$ which are closely related to the function $E_n(u)$ which we set out to estimate:

$$P(u,t)=\sum_{\nu=1}^{\infty}P_{\nu}(u)\frac{t^{2\nu}}{(2\nu-1)!}=\frac{\sinh \frac{1}{2}ut}{u\sinh \frac{1}{2}t}-\cosh\left(\frac{u^2-1}{12}\right)^{1/2}t = \frac{(u^2-1)(u^2-4)}{6!}\frac{t^4}{3!}+\cdots \tag{22}$$

where

$$P_{\nu}(u)=-\frac{1}{2\nu}\left(\frac{u^2-1}{12}\right)^{\nu}+\frac{1}{2\nu u}\left\{S_{2\nu}\left(\frac{1+u}{2}\right)-S_{2\nu}\left(\frac{1-u}{2}\right)\right\}. \tag{23}$$

We multiply both sides of (23) by $un^{-2\nu}$, and add, recalling that the odd powers of $1/n$ drop out of the expansion (17). This brings in the expression $(u/2)\ln(1-w^2/n^2)$ where $w^2=(u^2-1)/12$, which suggests the approximation (7). It gives for $E_n(u)$ the expression:

$$E_n(u)=u\sum_{\nu=1}^{\infty}P_{\nu}(u)n^{-2\nu} = \frac{u}{2}\ln\left(1+\frac{1-u^2}{12n^2}\right)-G\left(\frac{1+u}{2},\ n\right)+G\left(\frac{1-u}{2},\ n\right). \tag{24}$$

In like manner we obtain an exponential generating function $Q(u, t)$ for the polynomials $Q_{\nu}(u)$ which occur as coefficients of the powers of $1/n^2$ in the asymptotic expansion of the function $F_n(u)$ defined in (9). We have

$$F_n(u) = \sum_{\nu=2}^{\infty} Q_\nu(u) n^{4-2\nu} \tag{25}$$

where

$$\begin{aligned} Q(u, t) &= \sum_{\nu=1}^{\infty} Q_\nu(u) \frac{t^{2\nu}}{(2\nu-1)!} \\ &= \frac{6!}{(1-u^2)(4-u^2)} \left\{ \frac{\sinh \frac{1}{2}ut}{u \sinh \frac{1}{2}t} - \cosh \left(\frac{u^2-1}{12} \right)^{1/2} t \right\}. \end{aligned} \tag{26}$$

It is evident from (26) that the polynomials $Q_\nu(u)$ are even functions of u, and from (25) that $F_n(u)$ is an even function of both n and u. By expanding a few terms of (26) and substituting in (25) we find

$$\begin{aligned} F_n(u) = 1 &- \frac{85 - 25u^2}{126n^2} + \frac{107 - 40u^2 + 5u^4}{144n^4} \\ &- \frac{6467 - 2523u^2 + 405u^4 - 29u^6}{4752n^6} + \cdots . \end{aligned} \tag{27}$$

The coefficients appear somewhat simpler if $v = (1-u^2)/3$ is introduced as a new variable. Then

$$\begin{aligned} F_n(u) = f_n(v) = 1 &- \frac{20 + 25v}{42n^2} + \frac{8 + 10v + 5v^2}{16n^4} \\ &- \frac{160 + 200v + 106v^2 + 29v^3}{176n^6} + \cdots . \end{aligned} \tag{28}$$

The general expression for the polynomials $q_k(v)$ which occur as numerators of the terms in (28) is rather complicated. We state it here for completeness:

$$\begin{aligned} \frac{(-4)^k k q_k}{120} = &- \frac{v^{k-1} - (-1)^{k-1}}{3(v+1)} + \sum_{h=0}^{k-2} \frac{(4^h - 2)B_h}{2k+1} \binom{2k+1}{2h} \\ &\times \left\{ \frac{(3v-1)^{k-h} - (-1)^{k-h}}{3v} - \frac{(3v-1)^{k-h} - (-4)^{k-h}}{3(v+1)} \right\}. \end{aligned} \tag{29}$$

4. Computation of the correction factor $F_n(u)$. Although the terms in the alternating series (27) and (28) diminish rapidly when u is small and n is large, the νth term does not ultimately approach zero for fixed n as ν becomes infinite. Instead the series is an alternating asymptotic series, in which for fixed n sufficiently large the successive terms decrease at first and then increase without limit as ν becomes infinite.

For small n we cannot use the series (27) to evaluate the entries for $F_n(u)$ in table (10). Instead we make use of the Euler-MacLaurin summation formula applied to the function $1/n$:

$$\sum_{k=1}^{n} \frac{1}{k} = \gamma + \ln n + \frac{1}{2n} - \frac{B_1}{2n^2} + \frac{B_2}{4n^4} - \frac{B_3}{6n^6} + \cdots \tag{30}$$

$$\gamma = 0.57721\ 56649\ 01533 \cdots \text{ (Euler's constant).}$$

where B_ν are the Bernouilli numbers (14). We define the function $H(n)$ by the formula

$$H(n) = 120\left(\frac{B_2}{4} - \frac{B_3}{6n^2} + \frac{B_4}{8n^4} - \cdots\right) \tag{31}$$

$$= 120n^4\left\{\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2n} - \gamma - \ln n + \frac{1}{12n^2}\right\}.$$

Then

$$H(1) = 70 - 120\gamma = 0.73412020 \cdots, \tag{32}$$

$$H(2) = 0.90433,\ H(3) = 0.95229,\ H(4) = 0.97200.$$

In order to expand the generating functions (22) and (26) we need to use the fact that

$$\frac{t}{2} \operatorname{csch} \frac{t}{2} = 1 + \sum_{\nu=1}^{\infty} (1 - 2^{1-2\nu}) \frac{B_\nu}{(2\nu)!} (-t^2)^\nu. \tag{33}$$

Then by (26) and (22) we find

$$Q(0, t) = \frac{6!}{4} P(0, t) = \frac{6!}{4}\left\{\frac{t}{2} \operatorname{csch} \frac{t}{2} - \cos\frac{t}{(12)^{1/2}}\right\} \tag{34}$$

$$= \frac{6!}{4} \sum_{\nu=2}^{\infty}\left\{(1 - 2^{1-2\nu})B_\nu - \frac{1}{12^\nu}\right\} \frac{(-t^2)^\nu}{(2\nu)!}.$$

Hence, by (25), (26), and (34), we obtain

$$F_n(0) = \frac{6!}{4} \sum_{\nu=2}^{\infty}\left\{(1 - 2^{1-2\nu})B_\nu - \frac{1}{12^\nu}\right\} \frac{(-1)^\nu}{2\nu} n^{4-2\nu} \tag{35}$$

$$= \frac{3}{2} H(n) - \frac{3}{16} H(2n) - \frac{5}{8} (144n^4)\left\{\frac{1}{12n^2} - \ln\left(1 + \frac{1}{12n^2}\right)\right\}.$$

As an important special case we may calculate the smallest $F_n(0)$ for $n = 1$, namely

$$F_1(0) = 0.93180 - 0.31250 + 0.01736 - 0.00108 + 0.00007 = 0.63566. \tag{36}$$

Similarly, for $u = 1$, we find

$$Q(1, t) = \frac{6!}{6} P_n(1, t) = -120\left(\frac{t}{2}\coth\frac{t}{2} - 1 - \frac{t^2}{12}\right)$$

(37)

$$= 120\sum_{\nu=2}^{\infty} \frac{B_\nu(-t^2)^\nu}{(2\nu)!}$$

$$F_n(1) = 120n^4 \sum_{\nu=2}^{\infty} \frac{B_\nu}{2\nu} \frac{1}{(-n^2)^\nu} = H(n). \tag{38}$$

Thus the values of $F_n(1)$ are given by (32) and (38), those of $F_n(0)$ by (35), and the other values in table (10) by similar expansions. The data of table (10) may be fitted to at least two significant figures by the following rational function

$$F_n(u) \sim \frac{n^2 + 0.33 + 0.06(u^2 - 1)/n^2}{n^2 + 1 - 0.20u^2} \qquad n \geqq 1,\ 0 \leqq u \leqq 2. \tag{39}$$

By using this expression for $F_n(u)$ in conjunction with the formulas (8) and (9), we can evaluate the ratio of two values of the Gamma function to at least six significant figures, provided that the arguments differ by not more than 2.

A PROBLEM IN DIFFERENCE SETS

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1. Introduction. The propositions to be proved in this paper were suggested by a result quoted in *Scripta Mathematica** from an Italian journal.† They are substantially as follows: Let a, b, c, d be a set of 4 positive integers, and form the 4 absolute differences $|a-b|, |b-c|, |c-d|, |d-a|$ in a cyclical manner. The new set thus obtained will be called the derivative of the original set. If we now form, in the same manner, the higher derivatives of the original set, we shall, after a finite number of steps, arrive at a set all terms of which vanish.‡

The main object of this paper will be to show that this property does not depend on the fact that the terms of the original set are all integers. The precise theorem to be proved is:

THEOREM I: *Let a, b, c, d be a set of four real numbers. By forming the successive derivatives (as defined above) of this set we will always, after a finite number of steps, obtain a set all terms of which are equal to zero, except in the case in which the first derivative of the original set is, apart from trivial transformations, of the form*

$$1, q, q^2, q^3$$

* Scripta Mathematica, vol. V, p. 135 (April 1938).

† Periodiche di Matematiche, vol. 17, pp. 25–30 (1937).

‡ The same is true about sets of 8, 16, 32, · · · · integers (*ibidem*).

where q is the positive solution of the equation.

$$q^3 - q^2 - q - 1 = 0.$$

2. Discussion and proof of the theorem. Before entering upon the proof of Theorem I, we shall make a few general observations.

Consider $n(n \geqq 2)$ real numbers a_i arranged in a cyclical manner, *i.e.*, $\cdots a_n, a_1, a_2, \cdots, a_n, a_1 \cdots$. This arrangement will be called a cycle of n terms and denoted by Q_n. We shall write:

$$Q_n \equiv (a_1, a_2, \cdots, a_n), \quad \text{or} \quad Q_n \equiv (a_i).$$

It is immaterial which term of Q_n is regarded as the first; only the relative position of the terms in Q_n is important.

The cycle whose terms are the absolute differences of each two consecutive terms of Q_n, will be denoted by $Q_n' \equiv (a_i')$ and called the derivative of Q_n, *i.e.*,

$$Q_n' \equiv (|a_1 - a_2|, |a_2 - a_3|, \cdots, |a_{n-1} - a_n|, |a_n - a_1|).$$

The operation of finding the derivative will be called differentiation. The derivative of Q_n' will be denoted by Q_n'' and called the second derivative of Q_n, and so on. If after k (k finite) differentiations of Q_n we arrive at a $Q_n^{(k)}$ such that its terms are all zero, then Q_n will be called a vanishing cycle. If no such k exists, Q_n will be called a non-vanishing cycle.

If r consecutive terms are repeated in Q_n, m times in the same order, so that $n = m \cdot r$, $r \geqq 2$, $m \geqq 2$, Q_n will be called a periodic cycle; evidently all its derivatives are also periodic cycles.

When $n \geqq 3$ and the terms of Q_n form a geometric progression the last term of which equals the sum of all other terms, Q_n will be called a geometric cycle and denoted by Q_n^*, *i.e.*,

$$Q_n^* \equiv (a, aq, aq^2, \cdots, aq^{n-1}), \qquad a > 0, \qquad q^{n-1} = \sum_{i=0}^{n-2} q^i.$$

The last equation in q always has exactly one positive root,† $1 < q < 2$, increasing with n. The derivative $Q_n^{*\prime}$ is also a geometric cycle, since $a_i' = a_i(q-1)$ for all i (also for $i = n$, in view of the definition of q); the same is therefore true for all higher derivatives $Q_n^{*(k)}$. Hence we have the result: A geometric cycle never vanishes.

The terms a_i' of a derivative Q_n' of any cycle Q_n satisfy

$$\sum \pm a_i' = 0,$$

if we choose the proper signs. Indeed,

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† It is obvious that there is a root between 1 and 2. It follows from Descartes' rule of signs that there is only one positive root.

$$\sum \pm a_i' = + |a_1 - a_2| \pm |a_2 - a_3| \pm \cdots \pm |a_n - a_1|$$
$$= \pm [(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_1)] = 0.$$

Any cycle of two terms vanishes by no more than two differentiations. In fact

$$Q_2 \equiv (a, b); \quad Q_2' \equiv (|a - b|, |b - a|); \quad Q_2'' \equiv (0, 0).$$

Any Q_n, n odd, containing at least two unequal terms, never vanishes. This is seen as follows: If Q_n vanishes by exactly k differentiations, it is clear that $k>1$. Then $Q_n^{(k-1)}$ must contain only equal terms; but an odd number of equal terms cannot satisfy the condition $\sum \pm a_i^{(k)} = 0$, shown above to be necessary. Consequently $Q_n^{(k-1)}$ cannot be a derivative of any cycle, whence the contradiction.

For every n, except powers of 2, there exists an infinity of non-vanishing cycles Q_n. For odd n's, this has just been shown; if $n = 2^p \cdot r$, $r(\geqq 3)$ odd, at least all periodic cycles Q_n having 2^p periods of r unequal terms (and consequently all cycles having one such derivative) are non-vanishing.

If n is a power of 2, $(n = 2^p)$, there exist at least $p-1$ different non-vanishing cycles. These are the geometric cycle Q_n^* and the $p-2$ periodic cycles containing as periods one of the geometric cycles

$$Q_4^*, Q_8^*, \cdots, Q_{n/2}^*.$$

We shall now limit the discussion to cycles of 4 terms:

$$Q_4 \equiv Q \equiv (x, y, z, u).$$

LEMMA I: *If two non-consecutive terms of Q are equal, Q vanishes after not more than 4 differentiations.*

Proof: Let $z = x$;
then,

$$Q' \equiv (|x - y|, |x - y|, |x - u|, |x - u|),$$
$$Q'' \equiv (0, ||x - y| - |x - u||, 0, ||x - y| - |x - u||),$$

or

$$Q'' \equiv (0, t, 0, t),$$
$$Q^{(3)} \equiv (t, t, t, t),$$
$$Q^{(4)} \equiv (0, 0, 0, 0).$$

Now take any Q_4 and find Q'. The four terms of Q' may be written a, $a+b$, $a+b+c$, $a+b+c+d$, $(a, b, c, d \geqq 0)$; this of course is not necessarily their proper order in Q'. As the four terms of Q' must satisfy $\sum \pm a_i' = 0$, there arise two possibilities; namely,

I. the sum of two terms is equal to the sum of the rest, whence in all cases $d=b$.

II. the sum of three terms is equal to the fourth, and then $d=2a+b$.

Since we also have to take into account the relative position of the terms of Q', each case gives rise to three types:

A, in which a is not adjacent to $a+b$;
B, in which a is not adjacent to $a+b+c+d$;
C, in which a is not adjacent to $a+b+c$.

Thus Q' may be of one of the six following types:

	I	II
A	$(a, a+b+c, a+b, a+2b+c)$	$(a, a+b+c, a+b, 3a+2b+c)$
B	$(a, a+b, a+2b+c, a+b+c)$	$(a, a+b, 3a+2b+c, a+b+c)$
C	$(a, a+b, a+b+c, a+2b+c)$	$(a, a+b, a+b+c, 3a+2b+c)$

LEMMA II. *If Q' is not of the type* CII, *Q vanishes by at most seven differentiations.*

Proof: If Q' is of the types I, Q'' contains two equal non-adjacent terms and vanishes, by Lemma I, after 4 differentiations. If Q' is of the types AII or BII, then in Q'' the sum of two terms is equal to the sum of the other two, *i.e.*, Q'' is here of the types Q'I, and vanishes therefore after 5 differentiations.

COROLLARY: If in Q' one term is zero (*i.e.*, if Q contains two equal consecutive terms), Q vanishes by a few differentiations, as Q' is then of the types I.

By LEMMA II and its corollary all derivatives $Q_4^{(k)}$ of a non-vanishing cycle must be monotonic and of the form

$$(a, a+b, a+b+c, 3a+2b+c)$$

with a, b, c positive. In this case $Q_4^{(k+1)} \equiv (b, c, 2a+b, 2a+2b+c)$, and necessarily $b<c<2a+b$. Hence the monotony is preserved in the same sense in all derivatives. Denoting the first derivative of a non-vanishing Q_4 by P, and the higher ones by P_1, P_2, P_3, and so on, P may be written

$$(x, y, z, x+y+z), \text{ where } 0 < x < y < z,$$

and similarly

$$P_n \equiv (x_n, y_n, z_n, x_n+y_n+z_n), \qquad 0 < x_n < y_n < z_n.$$

But

$$P_1 \equiv (y-x, z-y, x+y, y+z).$$

Denoting by M the matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

we may write:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and generally:

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = M^n \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Here are the values of some M^n which will be needed later:

$$M^2 = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \qquad M^3 = 2 \cdot \begin{pmatrix} 0 & 2 & -1 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$M^5 = 2 \cdot \begin{pmatrix} 3 & 4 & -3 \\ -3 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}, \qquad M^6 = (M^3)^2 = 2^2 \cdot \begin{pmatrix} -3 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix},$$

$$M^{12} = (M^6)^2 = 2^4 \cdot \begin{pmatrix} 5 & 12 & -8 \\ -8 & -3 & 4 \\ 4 & -4 & 1 \end{pmatrix}, \quad M^{24} = (M^{12})^2 = 2^8 \cdot \begin{pmatrix} -103 & 56 & 0 \\ 0 & -103 & 56 \\ 56 & 56 & -47 \end{pmatrix};$$

and generally, for $k = 0, 1, 2, \cdots$,

$$M^{3 \cdot 2^k} = 2^{2^k} \cdot T_k, \quad \text{where} \quad T_k = \begin{pmatrix} a_{1,k} & b_{1,k} & c_{1,k} \\ a_{2,k} & b_{2,k} & c_{2,k} \\ a_{3,k} & b_{3,k} & c_{3,k} \end{pmatrix};$$

the numbers $a_{i,k}$, $b_{i,k}$, $c_{i,k}$ are integers.

By the preceding,

$$0 < y_6 = 4(2x - y), \quad 0 < y_8 = 2(-3x + z), \quad 0 < z_8 = 4(2y - z).$$

Hence, $y < 2x$, $3x < z < 2y$, and therefore

$$x_1 = y - x < x, \qquad y_1 = z - y < y, \qquad z_1 = x + y < 3x < z.$$

Thus the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are strictly descending, and clearly $0<x_{24}<x$, $0<y_{24}<y$. But

$$x_{24} = 2^8(-103x + 56y), \qquad y_{24} = 2^8(-103y + 56z);$$

hence, $1\cdot8392<(y/x)<1\cdot8394$, $1\cdot8392<(z/y)<1\cdot8394$. From this we easily deduce that the ratios x_1/x, y_1/y, z_1/z are less than $0\cdot84$, and therefore less than $2^{-1/4}$. The same is true generally about x_{k+1}/x_k, *etc.* Hence, x_n/x, y_n/y, z_n/z are all less than $2^{-n/4}$.

Now consider the infinite set S of derivatives P_n, where $n=3\cdot2^k$. As the determinant of M is

$$|M| = \begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2,$$

it follows that $|M^n| = 2^n$ and $|I_k| = 1$; consequently in every $|T_k|$ at least one minor is different from zero. Hence S contains an infinite sub-set S_1, for the terms of which at least one minor, of constant place in T_k, always differs from zero. We shall assume that in S_1, $D(a_{3,k})$, the minor of $s_{3,k}$, always differs from zero. Later on it will become clear that this assumption does not affect the generality of our proof.

For the terms x_n and y_n of every P_n belonging to S_1 we may write

$$0 < x_n < (2^{-1/4})^{3\cdot 2^k}\cdot x = x\cdot 2^{-3\cdot 2^{k-2}},$$
$$0 < y_n < 2x_n < 2x\cdot 2^{-3\cdot 2^{k-2}},$$

or

$$0 < 2^{2^k}(a_1x + b_1y + c_1z) < x\cdot 2^{-3\cdot 2^{k-2}},$$
$$0 < 2^{2^k}(a_2x + b_2y + c_2z) < 2x\cdot 2^{-3\cdot 2^{k-2}},$$

(writing simply a_i, b_i, c_i instead of $a_{i,k}$, $b_{i,k}$, $c_{i,k}$).

We now write

$$\begin{cases} a_1 + b_1\left(\dfrac{y}{x}\right) + c_1\left(\dfrac{z}{x}\right) = \delta_k, \\ a_2 + b_2\left(\dfrac{y}{x}\right) + c_2\left(\dfrac{z}{x}\right) = \epsilon_k, \end{cases}$$

where δ_k and ϵ_k denote positive numbers which are arbitrarily small for sufficiently large k; indeed,

$$0 < \delta_k < 2^{-7\cdot 2^{k-2}}, \qquad 0 < \epsilon_k < 2\cdot 2^{-7\cdot 2^{k-2}}.$$

(For a definite k, δ_k and ϵ_k may vary perhaps, but at any rate not outside the given intervals.)

Solving the two equations for y/x and z/x, we obtain:

$$\frac{y}{x} = \frac{c_1a_2 - c_2a_1}{b_1c_2 - b_2c_1} + \eta_k, \quad \text{where} \quad \eta_k = \frac{\delta_k c_2 - \epsilon_k c_1}{b_1c_2 - b_2c_1}.$$

The denominators are $D(a_{3,k})$, which by hypothesis is not zero; as a determinant of integers, $|D(a_{3,k})| = |b_1c_2 - b_2c_1| \geqq 1$, and

$$|\eta_k| \leqq |\delta_k c_2 - \epsilon_k c_1| \leqq \delta_k |c_2| + \epsilon_k |c_1| \leqq (\delta_k + \epsilon_k)m_k,$$

denoting by m_k the maximum of $|a_{i,k}|$, $|b_{i,k}|$, $|c_{i,k}|$. As $T_{k+1} = (T_k)^2$, the terms of T_{k+1} are of the form:

$$a_{i,k}j_{1,k} + b_{i,k}j_{2,k} + c_{i,k}j_{3,k},$$

i standing for 1, 2 or 3, and j standing for a, b or c. Hence $m_{k+1} \leqq 3(m_k)^2 = \frac{1}{3}(3m_k)^2$, and, by induction,

$$m_{k+p} \leqq \tfrac{1}{3}(3m_k)^{2^p}.$$

For $k=2$, *i.e.*, $n = 3 \cdot 2^2 = 12$, we found above (see M^{12}) $m_k = 12$; therefore, for $k > 2$,

$$m_k \leqq \tfrac{1}{3}(3 \cdot 12)^{2^{k-2}},$$

and

$$|\eta_k| \leqq (\delta_k + \epsilon_k)m_k < 3 \cdot 2^{-7 \cdot 2^{k-2}} \cdot \tfrac{1}{3} \cdot (36)^{2^{k-2}} = \left(\frac{9}{32}\right)^{2^{k-2}}.$$

Consequently $|\eta_k|$ may be as small as we please, and, as it is the absolute value of the difference between y/x and $(c_1a_2 - c_2a_1)/(b_1c_2 - b_2c_1)$, it follows that y/x must lie inside an interval shorter than

$$\theta_k = 2 \cdot \left(\frac{9}{32}\right)^{2^{k-2}}.$$

But every such interval must contain the real number q, the root of the cubic equation $q^3 - q^2 - q - 1 = 0$. Since θ_k tends to zero as $k \to \infty$, the sequence of the intervals of length θ_k cannot contain more than one common number. Thus $y/x \equiv q$, and similarly it is shown that $z/x \equiv q^2$.

This completes the proof of Theorem I.

Note: By further consideration it may now be deduced that, in order to be a non-vanishing cycle, Q_4 must have one of the two forms:

$$(u,\ u + x,\ u + x + xq,\ u + xq^2),$$

or

$$(u + xq^3,\ u + xq + xq^2,\ u + xq^2,\ u),$$

in which u is arbitrary, $x \neq 0$, $q^3 - q^2 - q - 1 = 0$, q real. (Since only then $Q_4' \equiv Q_4^*$.)

THE RATIONAL CANONICAL FORM OF A MATRIX II

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1. Introduction. In a previous paper* we attempted a direct attack on the problem of the reduction of a square matrix A with elements in a field K to rational canonical form. Our result was, in a sense, inconclusive, since we used the theory of invariant factors to complete our reduction. MacDuffee's Carus Monograph, *Vectors and Matrices*† has since appeared. In this book an elegant reduction is obtained. However, the present writer feels that the theory of the derogatory case (VM, pp. 128–136) might well offer difficulty to a beginner. For this reason we shall present in this note a completion of the reduction process of RC which is independent of invariant factors, and which does not involve a special consideration of the derogatory case. In effect, we adapt one of MacDuffee's results to the stage of reduction achieved in RC by means of elementary similarity transformations.

Our notation and terminology will be that of VM with the additional designation of the elementary similarity transformation $p_{ij}(c)$ for $c \epsilon K$ of RC. We shall find it convenient to denote the *companion matrix* (VM, p. 82) of a non-constant monic polynomial $f(x) \epsilon K[x]$ by $C(f)$.

2. Two basic theorems. In this section we shall give the additional theory needed to complete our reduction. The reader will observe that, aside from a bare minimum of the theory of matric polynomials, we employ only the basic Hamilton-Cayley theorem.

THEOREM 1. *If $f(x)$ and $g(x)$ are non-constant monic polynomials of $K[x]$, then the matrix $C(fg)$ is similar in K to*

$$B = \begin{bmatrix} C(f) & 0 \\ U & C(g) \end{bmatrix} \tag{1}$$

where U is a matrix with unity in the upper right-hand corner and zeros elsewhere.

Proof. Let $f(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$, and let $g(x)$ have degree m. Then the sequence of elementary similarity transformations: $p_{m+n,\, m+n-1}(-a_{n-1})$, $p_{m+n,\, m+n-2}(-a_{n-2})$, $\cdots$, $p_{m+n,\, m}(-a_0)$; $p_{m+n-1,\, m+n-2}(-a_{n-1})$, $p_{m+n-1,\, m+n-3}(-a_{n-2})$, $\cdots$, $p_{m+n-1,\, m-1}(-a_0)$; $\cdots$; $p_{n+1,\, n}(-a_{n-1})$, $\cdots$, $p_{n+1,\, 1}(-a_0)$, yields B.

Note that if g factors in K, our method applied to $C(g)$ leaves U unaltered.

THEOREM 2. *Let $f(x)$ and $g(x)$ be non-constant monic polynomials of $K[x]$ which satisfy $a(x)f(x) + b(x)g(x) = 1$ for polynomials $a(x)$, $b(x) \epsilon K[x]$. Define $P = b(B)g(B) + (0 + I)$, where B is given by (1) and the order of I is the degree of $g(x)$. Then P is non-singular and $P^{-1}BP = C(f) + C(g)$.*

* The rational canonical form of a matrix, this MONTHLY, vol. 49, 1942, pp. 451–454. We shall refer to this paper as RC.

† We shall refer to this book as VM.

Proof. The Hamilton-Cayley theorem yields the matric equation

$$b(C(f))g(C(f)) = I. \tag{2}$$

By induction we easily see that

$$B^i = \begin{bmatrix} (C(f))^i & 0 \\ X_i & (C(g))^i \end{bmatrix} \qquad (i = 1, 2, \cdots) \tag{3}$$

in which X_i is a matrix which will not concern us. By (2), (3), and the Hamilton-Cayley theorem, we find that

$$b(B)g(B) = \begin{bmatrix} b(C(f)) & 0 \\ Y & b(C(g)) \end{bmatrix}\begin{bmatrix} g(C(f)) & 0 \\ Z & g(C(g)) \end{bmatrix} = \begin{bmatrix} I & 0 \\ W & 0 \end{bmatrix}.$$

The equation $Bb(B)g(B) = b(B)g(B)B$ then gives $WC(f) = U + C(g)W$. A direct matrix calculation now justifies the conclusion of Theorem 2.

Remark. It will be noted that no use was made of the particular form of the matrix U in the proof of Theorem 2.

3. Completion of the reduction process. We now apply the theorems of Section 2 to complete the reduction of the matrix A which was initiated in *RC*. We may assume (by applying the process of *RC* to A^T and transposing the result) that A is similar in K to $C(f_1) + C(f_2) + \cdots + C(f_t)$, where $f_i(x) \epsilon K[x]$ for $i = 1, 2, \cdots, t$. We may then factor each f_i into irreducible factors in $K[x]$ and apply Theorem 1 (noting the remark that follows it) and then Theorem 2 to find that $C(f_i)$ is similar in K to $C(g_{i1}) + \cdots + C(g_{is_i})$ with each $g_{ij} (j = 1, 2, \cdots, s_i)$ a power of an irreducible polynomial of $K[x]$ and with the g_{ij} ($j = 1, 2, \cdots, s_i$; i fixed) relatively prime in pairs. Another application of Theorem 1 gives the desired canonical form. We should emphasize that the usual bookkeeping scheme* for recording a sequence of elementary transformations permits a simultaneous calculation of the linear transformation which reduces A.

4. An example. This section is devoted to an illustration of the use of the theorems of Section 2 in a very simple example.

Let us consider the reduction of $C = C(x^3 - x^2 - x + 1)$. Suppose first that the characteristic of K is not two. We then have $x^3 - x^2 - x + 1 = (x+1)(x-1)^2$ and Theorem 1 yields

$$B = Q^{-1}CQ = \begin{bmatrix} C((x-1)^2) & 0 \\ U & C(x+1) \end{bmatrix}$$

where

* See, for example, G. Birkhoff and S. MacLane, A Survey of Modern Algebra, p. 276, or A. A. Albert, Introduction to Algebraic Theories, p. 42.

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $a(x)(x-1)^2+b(x)(x+1)=1$ with $4a(x)=1$ and $4b(x)=3-x$. Compute

$$4b(B)g(B) = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

and

$$4P = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ -1 & 1 & 4 \end{bmatrix}, \qquad 4P^{-1} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix}.$$

Then $P^{-1}BP=C((x-1)^2)+C(x+1)$. Using Theorem 1 again yields

$$R^{-1}P^{-1}BPR = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{with} \quad R = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If the characteristic of K is two, then $x^3-x^2-x+1=(x-1)^3$ and we obtain our canonical form directly from Theorem 1 by means of

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{for we have} \quad S^{-1}CS = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

5. Concluding remarks. We cannot close this short note without a few brief remarks which our method suggests. It is not our feeling that the general theorems on the relation of a matrix to its invariant spaces and on the characteristic and minimum functions of a matrix should be suppressed. We are as heartily in favor of these "by-products" as is MacDuffee (VM, p. vii). It should be observed, however, that the proofs of these theorems become almost trivial once the rational canonical form is obtained.

MATHEMATICAL NOTES

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SELF-DUAL FRAGMENTS OF THE ORDINARY PLANE

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In the geometry of our ordinary plane, points and lines play different rôles. There exist parallel lines, that is, lines which are not on a common point. In fact, for any given line l, there exists exactly one parallel line on every point which is not on l. But there are no parallel points, in the sense of points which are not on a common line. Every two distinct points are on exactly one common line.

The system of all points (x, y) and all lines $y=mx+c$ (that is, all lines which are not parallel to the Y-axis), however, is perfectly self-dual. That is to say, in this system points and lines do play precisely the same role. We shall call two points which are not on a line belonging to the system, *vertical* points. We call (m, c) the *non-vertical line coordinates* of the line $y=mx+c$, and prove a perfect analogy between m and x, and between c and y. We shall accentuate the analogy by denoting* the slope of a line by X, and its y-intercept by $-Y$. Then the point (x, y) is on the line (X, Y) if and only if $y=Xx-Y$, that is,

$$xX = y + Y.$$

Clearly, we can say:

1. In order that the two lines (X_1, Y_1) and (X_2, Y_2) be parallel it is necessary and sufficient that $X_1=X_2$. And dually: In order that the two points (x_1, y_1) and (x_2, y_2) be vertical it is necessary and sufficient that $x_1=x_2$.

2. If (x, y) is a given point, and (X, Y) is a given line, then there exists one and only one line on (x, y) which is parallel to (X, Y), namely the line $(X, y-xX)$. The dual counterpart of this euclidean parallel postulate reads: If (X, Y) is a given line, and (x, y) is a given point, then there exists one and only one point on (X, Y) which is vertical to (x, y), namely, the point $(x, Y-xX)$.

Starting with the self-dual projective plane we obtain the ordinary plane by deleting a line and all points on it. The result is, of course, not self-dual. We obtain our self-dual system by deleting a line, l_∞, and all points on l_∞, as well as a point, P_∞, and all lines on P_∞—with the proviso that P_∞ is on l_∞. Clearly, the result is again self-dual.

If in the above procedure we delete a point, P_0, which is not on the deleted line, l_∞, then we obtain another self-dual fragment of our ordinary plane. If we

* This notation was suggested by H. S. M. Coxeter.

choose P_0 as the origin (0, 0), then the new system consists of all points $(x, y) \neq (0, 0)$ and all lines $(a, b) \neq (0, 0)$ such that (x, y) is on (a, b) if and only if $ax+by=1$. In this theory, two points are not on a common line or, as we might say, *proportional* if and only if they are collinear with the origin. Again, lines and points satisfy the same postulates. Both, however, satisfy the following non-euclidean parallel postulates:

(a) Given a point P, and a line l, there is at most one line on P which is parallel to l, and at most one point on l, which is proportional to P.

But there is no line parallel to the line (a, b) on a point (x, y) for which $ax+by=0$. Similarly, there is no point proportional to the point (x, y) on the line (a, b) for which $ax+by=0$. All we can say is:

(b) If two points are not on a line parallel to l, then the two points are proportional; and if two lines are not on a point proportional to P, then the two lines are parallel.

We obtain a self-dual fragment of our ordinary space by deleting from the projective space a plane p_∞, and all points and lines on p_∞, as well as a point, P_∞, and all planes and lines on P_∞—with the proviso that P_∞ is on p_∞. We retain the planes $z=ax+by+c$ or $z=Xx+Yy-Z$. The point (x, y, z) and the plane (X, Y, Z) are incident if and only if $xX+yY=z+Z$. This fragment satisfies Euclid's parallel postulate and the dual statement.

Also the theory of order including the Law of Pasch is capable of dualization.

ON THE FEUERBACH POINTS*

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Let ABC be a triangle with circumcenter O; (I) the incircle with center I and D, E, F as points of contact with the sides BC, CA, AB; G the centroid; D', E', F' the points of intersection of the line $d \equiv OI$ with the sides EF, FD, DE of the triangle DEF; and Z the point of contact of the nine-point circle with the incircle (the *in-Feuerbach point*).

THEOREM 1. *The Miquel circle of the complete quadrilateral $Q \equiv (DEF, d)$ passes through the in-Feuerbach point.*

The lines DD_1, EE_1, FF_1 which join the points D, E, F with the circumcenters D_1, E_1, F_1 of the triangles $DE'F'$, $EF'D'$, $FD'E'$ of the quadrilateral Q, being the isogonal conjugates for the angles D, E, F of the altitudes DD'', EE'', FF'' of the above triangles, are concurrent on the circle (I) at the focus of the parabola inscribed in the triangle DEF and having directrix d, that is to say, at the in-Feuerbach point Z of triangle ABC [1]. On the other hand, since triangle $D_1E_1F_1$ is directly similar to triangle DEF, angles EZD and $E_1F_1D_1$ are

* Translated from the French by Howard Eves.

equal or supplementary, and the circle (I_1) circumscribed about triangle $D_1E_1F_1$ (the *Miquel circle of* Q) passes through Z.

COROLLARY 1.1. *The line ZM joining the in-Feuerbach point Z to the Miquel point M of Q coincides with the radical axis of the circles (I) and (I_1).*

COROLLARY 1.2. *The points Z and M are symmetric with respect to the line IG.*

This interesting property is easily established with the aid of complex coordinates.

THEOREM 2. *If A_1, B_1, C_1 are the orthogonal projections of the circumcenter on the interior bisectors AI, BI, CI, then parallels through D, E, F to the lines B_1C_1, C_1A_1, A_1B_1, respectively, are concurrent at the in-Feuerbach point.*

The points A_1, B_1, C_1 are located on the circle (I') described on OI as diameter, the lines IA_1, IB_1, IC_1 are perpendicular to the sides EF, FD, DE of triangle DEF, and the lines OA_1, OB_1, OC_1 are parallel to these same sides. As a result we see that

$$\sphericalangle C_1B_1O = \sphericalangle C_1IO = \sphericalangle F'DD'' = \sphericalangle ZDF,$$

and consequently that lines C_1B_1 and ZD are parallel, and similarly for B_1A_1 and ZF, A_1C_1 and ZE.

We have extended these two constructions of the in-Feuerbach point, which also apply with obvious modification to the three outer-Feuerbach points, to the construction of the orthopole of any diameter of the circumcircle of the fundamental triangle [2].

1. V. Thébault, Journal de Vuibert, 36e année, 1911, p. 2.
2. V. Thébault, Comptes-Rendus de l'Académie des Science (Paris), 1944, vol. 218, p. 434

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All materials for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.

A NOTE ON A METHOD OF LORD RAYLEIGH

L. M. COURT, Rutgers University

A familiar result in the theory of linear differential equations, associated with Lagrange's name, states that if n linearly independent solutions of the associated homogeneous (reduced) equation are known, a particular integral of any nth order linear differential equation can be found by a simple quadrature.

The integrand in this case is the quotient of two determinants in the n linearly independent solutions (the determinant in the denominator is a Wronskian) multiplied by the function on the right side of the non-homogeneous equation. When $n=2$, all this reduces to the result that if $u_1(t)$ and $u_2(t)$ are two linearly independent solutions of $u''+p(t)u'+q(t)=0$, then a particular integral of the non-homogeneous equation

$$u'' + p(t)u' + q(t) = F(t) \tag{1}$$

is furnished by the expression

$$u(t) = \int_0^t \frac{\begin{vmatrix} u_1(\lambda) & u_1(t) \\ u_2(\lambda) & u_2(t) \end{vmatrix}}{D(\lambda)} F(\lambda)\,d\lambda, \tag{2}$$

where $D(\lambda)$ is the Wronskian $u_1(\lambda)\,u_2'(\lambda)-u_1'(\lambda)\,u_2(\lambda)$.

The method of obtaining (2) found in most texts, as for example Ince, is purely analytic. But for the special case in which p and q are constants, Lord Rayleigh was able to get (2) by a mechanical "strategem" which, if not nearly as rigorous, is much more intuitive. We shall retrace Rayleigh's reasoning* somewhat more fully than he does himself, and then go on to show that it can be extended to the general case, *i.e.*, equation (2).

When p and q are constants and, in addition, $4q-p^2$ is positive, (2) represents mechanically a particle of unit mass "vibrating" about its position of equilibrium $u=0$ under the influence of the impressed force $F(t)$ and against a frictional resistance proportional to the velocity. When we put $F(t)\equiv 0$, we "disconnect" the impressed force and "free" the system. After solving the homogeneous equation representing the free system, we can express the arbitrary constants in terms of the particle's position $u(\lambda)$ and velocity $u'(\lambda)$ at the time λ:

$$u(t) = \frac{e^{p(\lambda-t)/2}}{m}\left\{u'(\lambda)\sin m(t-\lambda) + u(\lambda)\left[\frac{p}{2}\sin m(t-\lambda) + m\cos m(t-\lambda)\right]\right\} \tag{3}$$

$$\left(\text{where } m^2 = q - \frac{p^2}{4} > 0\right).$$

The particle's position at any time t is thus seen to depend parametrically upon its position at the time λ and, what is more important for our purposes, its velocity at the time λ; both of these are, of course, arbitrary.

Now let us "connect" the impressed force with the system at the instant $\lambda-d\lambda$, permitting it to operate for a brief time interval $d\lambda$. The impressed force

* Lord Rayleigh, *The Theory of Sound*, Vol. 1, p. 62.

$F(t)$ generates a velocity increment $F(\lambda)d\lambda$ in our particle of unit mass, so that its velocity at the time λ is no longer $u'(\lambda)$, as before, but $u'(\lambda)+F(\lambda)d\lambda$. Because of this change in its "initial" velocity at λ, the particle's position at any later instant t will differ from what it would have been if the impressed force had never operated; in fact, referring back to (3), we see that this deviation is given by

$$\frac{e^{p(\lambda-t)/2}}{m}\,[F(\lambda)d\lambda]\,\sin m(t-\lambda). \tag{4}$$

If now we permit the impressed force to operate continuously from $\lambda=0$ to $\lambda=t$, since displacements along the same straight line are algebraically additive, we see that the total deviation in the particle's position from what it otherwise would have been is given by the integral

$$\frac{1}{m}\int_0^t e^{p(\lambda-t)/2}\sin m(t-\lambda)F(\lambda)d\lambda. \tag{5}$$

But (5) when added to (3) gives the particle's position at any instant t, the mechanical system, of course, having been subjected to the impressed force $F(t)$ all the time. Accordingly, this sum is a solution (the complete solution since it involves two arbitrary constants) of the differential equation representing the "forced" system. That is, (5) is a particular integral of the non-homogeneous equation (1) for p and q constant, with the further proviso that $4q-p^2>0$.

Let us now generalize Rayleigh's reasoning to the case in which p and q are not constants. When p and q are functions of t, (1) no longer represents a "vibrating" system but a general system, *i.e.*, a system consisting of a particle of unit mass moving in a more or less general fashion along a straight line under the influence of the impressed force $F(t)$ and against a frictional resistance. As before, the complete solution of the associated homogeneous equation

$$u(t) = \alpha u_1(t) + \beta u_2(t) \tag{6}$$

represents the motion (position) of the particle when $F(t)$ is "disconnected." By solving a pair of algebraic equations, the arbitrary constants α and β can be expressed in terms of the particle's position $u(\lambda)$ and velocity $u'(\lambda)$ at the time λ, and (6) rewritten as

$$u(t) = \frac{1}{D(\lambda)}\left\{u'(\lambda)\begin{vmatrix} u_1(\lambda) & u_1(t) \\ u_2(\lambda) & u_2(t)\end{vmatrix} - u(\lambda)\begin{vmatrix} u_1'(\lambda) & u_1(t) \\ u_2'(\lambda) & u_2(t)\end{vmatrix}\right\}, \tag{7}$$

where $D(\lambda)$ is once again the Wronskian $u_1(\lambda)\,u_2'(\lambda)-u_1'(\lambda)\,u_2(\lambda)$. If $F(t)$ is brought into play at the instant $\lambda-d\lambda$ and permitted to operate for a time interval $d\lambda$, the particle's velocity at the instant λ will be $F(\lambda)d\lambda$ greater than it would have been otherwise; its position at any subsequent instant t will deviate from what its position would have been otherwise by an amount

$$F(\lambda)d\lambda \frac{1}{D(\lambda)} \begin{vmatrix} u_1(\lambda) & u_1(t) \\ u_2(\lambda) & u_2(t) \end{vmatrix};$$

finally, its total displacement in position if $F(\lambda)$ is permitted to operate continuously from $\lambda=0$ to $\lambda=t$ will be given by the integral

$$\int_0^t \frac{\begin{vmatrix} u_1(\lambda) & u_1(t) \\ u_2(\lambda) & u_2(t) \end{vmatrix}}{D(\lambda)} F(\lambda)d\lambda, \tag{8}$$

and this, which is identical with (2), must be a particular integral of (1) even when p and q are general functions of the time.

It is fairly obvious that this method can be extended to the general linear differential equation of the nth order. In this case, after solving the associated homogeneous equation, we express the n arbitrary constants of integration in terms of the derivatives $u^{(0)}(t) \equiv u(t)$, $u^{(1)}(t)$, $u^{(2)}(t)$, $\cdots$, $u^{(n-1)}(t)$ evaluated at the time λ. The first three of these have familiar mechanical names; there are no standard designations for the remaining $n-3$ but it is evident that they are replete with mechanical significance, particularly when the acceleration is not constant but varies in a complicated fashion with the time. We might term $u^{(n-1)}(\lambda)$ the particle's *velocity of the* $(n-1)$*th order*. $F(\lambda)$ will be the *impressed force of the* $(n-1)$*th order*, *i.e.*, if it operates for a brief interval $d\lambda$, it generates an increment $F(\lambda)\,d\lambda$ in the velocity of the $(n-1)$th order of a particle of unit mass. The quantity that multiplies $u^{(n-1)}(\lambda)$ after we have expressed the n original arbitrary constants in terms of $u^{(0)}(\lambda)$, $\cdots$, $u^{(n-1)}(\lambda)$ will be the quotient of two determinants of the nth order, the numerator being a function of λ and t and the denominator the Wronskian $D(\lambda)$ of the n linearly independent solutions $u_1(\lambda)$, $\cdots$, $u_n(\lambda)$ of the associated homogeneous equation. If we permit the impressed force of the $(n-1)$th order $F(\lambda)$ to operate continuously from $\lambda=0$ to $\lambda=t$, we see that the deviation in the particle's position will be given by an integral similar to (8)—except that the two determinants are of the nth order—or, in other words, by the known expression for the particular integral in terms of the n linearly independent solutions of the reduced equation.

THE LAWS OF SINES AND COSINES

L. J. Burton, Bryn Mawr College

Using the distance formula of analytic geometry we may derive the law of cosines in the following simple fashion. For any triangle ABC, choose the coordinate system so that A is at the origin and B is on the positive x-axis. Then B is $(c, 0)$ and C is $(b \cos A, b \sin A)$. From the distance formula:

$$a^2 = (b \cos A - c)^2 + (b \sin A)^2 = b^2 + c^2 - 2bc \cos A.$$

Most textbooks do not point out any connection between the law of cosines

and the law of sines. Assuming the former we can prove the latter algebraically as follows:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{and} \quad \sin^2 A = 1 - \cos^2 A = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4b^2c^2}$$

$$\text{or} \qquad \sin^2 A = \frac{(a + b - c)(a - b + c)(a + b + c)(-a + b + c)}{4b^2c^2}.$$

By symmetry:

$$\sin^2 B = \frac{(a + b - c)(-a + b + c)(a + b + c)(a - b + c)}{4a^2c^2}.$$

Then $b^2 \sin^2 A = a^2 \sin^2 B$; and since $\sin A, \sin B \geqq 0$, we have

$$b \sin A = a \sin B.$$

Also, it follows that if $s = \frac{1}{2}(a+b+c)$, then

$$\sin A = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}$$

and since the area of the triangle ABC is clearly $\frac{1}{2}bc \sin A$, we have a simple proof that the area is $\sqrt{s(s-a)\,(s-b)\,(s-c)}$.

The law of cosines can be proved algebraically from the law of sines as follows. Assuming that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

we wish to prove that $a^2 = b^2 + c^2 - 2bc \cos A$, or simply the trigonometric identity

$$\sin^2 A = \sin^2 B + \sin^2 C - 2 \sin B \sin C \cos A$$

where $A = \pi - (B+C)$. Putting $\sin A = \sin B \cos C + \cos B \sin C$ and $\cos A = \sin B \sin C - \cos B \cos C$, the identity is easily verified. In spherical trigonometry, the law of sines follows from the law of cosines by a proof similar to the above, but the law of cosines does not follow from the law of sines.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 881. *Proposed by C. M. Sandwick, Easton High School, Easton, Pa.*

(a) Show that any positive integer can be represented uniquely in the form $\sum_{n=1}^{\infty} a_n n!$, $0 \leqq a_n \leqq n$.

(b) Show that any positive rational number less than unity can be represented uniquely in the form $\sum_{n=1}^{\infty} a_n/(n+1)!$, $0 \leqq a_n \leqq n$.

E 882. *Proposed by C. W. Trigg, Los Angeles City College*

Five regular tetrahedra arranged around a common edge just fail to fill space around that edge. (a) What is the closest approximation to a regular tetrahedron such that five such tetrahedra will fill the space around a common edge to form a decahedron having equilateral faces? (b) Show that the edge of the decahedron is $\sqrt{5}$ times the radius of the sphere touching those edges which radiate from the axis of the decahedron.

E 883. *Proposed by H. T. R. Aude, Colgate University*

Find the smallest integer n which is the sum of the squares of two non-negative integers in exactly (a) three ways, (b) five ways, (c) six ways.

E 884. *Proposed by E. P. Starke, Rutgers University*

Show that there exist infinitely many rational integral equations with integral coefficients and leading coefficient unity, and of degree n, such that $n-1$ of the roots occur within a specified interval, however small.

E 885. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In the tetrahedron $ABCD$ let A_1, B_1, C_1, D_1 divide a set of concurrent cevians AA', BB', CC', DD' in the same ratio, and let A_1B_1, A_1C_1, A_1D_1 pierce BCD in A_2, A_3, A_4. Show that triangles A_2CD, A_3DB, A_4BC have equal areas.

SOLUTIONS

Cevian Triangles of Incenter and Excenters

E 849 [1949, 31]. *Proposed by C. W. Trigg, Los Angeles City College*

The area of a triangle is to the area of the triangle determined by the points of contact of its incircle (or excircle) as its circumdiameter is to its inradius (or

exradius).

Solution by C. S. Ogilvy, Trinity College. Let K and K' denote the areas of the given triangle and the cevian triangle of the incenter. Then, in terms of standard notation,

$$K/K' = r(a + b + c)/r^2(\sin A + \sin B + \sin C) = 2R/r.$$

The same procedure, with due regard to signs, yields the corresponding theorem for an excenter.

Also solved by Roger Lessard, Azriel Rosenfeld, N. C. Scholomiti, W. R. Talbot, Kaidy Tan, P. D. Thomas, Roscoe Woods, and the proposer.

Tan located the problem in Loney, *Plane Trigonometry*, p. 248, and Joseph Langr pointed out that the problem was studied by Neuberg in *Mathesis*, 1910, p. 258. An easy secondary result, noted by the proposer, is that the areas of the cevian triangles of the incenter and the excenters are to each other as the radii of the corresponding circles.

Area of Quadrilateral with Incircle and Circumcircle

E 851 [1949, 104]. *Proposed by Joseph Rosenbaum, Hartford, Connecticut*

The area of a quadrilateral which has both a circumcircle and an incircle is equal to the square root of the product of its sides.

I. *Solution by M. C. Stapp, University of Alabama.* Let $ABCD$ be the quadrilateral, O the incenter, r the inradius, and P, Q, R, S the points of contact of the incircle with the sides AB, BC, CD, DA. Set $t = AS = AP$, $u = BP = BQ$, $v = CQ = CR$, $w = DR = DS$. Since $ABCD$ is concyclic, right triangles OQC and APO are similar, whence $r^2 = tv$. Similarly $r^2 = uw$, and we have $tv = uw$. If K is the sought area we then have

$$\begin{aligned} K^2 &= r^2(t + u + v + w)^2 = uw(t + u + v + w)^2 \\ &= uw(uw/v + u + v + w)^2 = uw(t + u + uw/t + w)^2. \end{aligned}$$

Therefore, taking the product of the last two expressions,

$$\begin{aligned} K^4 &= (uw/tv)^2(uw + vu + v^2 + vw)^2(t^2 + tu + uw + tw)^2 \\ &= (v + w)^2(v + u)^2(t + u)^2(t + w)^2 \\ &= (AB)^2(BC)^2(CD)^2(DA)^2. \end{aligned}$$

II. *Solution by L. R. Chase, Newport, R. I.* Let a, b, c, d be the sides, in their order. Since the quadrilateral is a circumscribed one, $a + c = b + d = s$. Since the quadrilateral is an inscribed one, the area is

$$K = [(s - a)(s - b)(s - c)(s - d)]^{1/2},$$

which, by appropriate substitutions for s, becomes $(cdab)^{1/2}$.

Also solved by Louis Berkofsky, W. C. Bornmann, W. E. Buker, William Douglas, Ragnar Dybvik, W. O. Pennell, C. M. Sandwick, N. C. Scholomiti,

C. W. Trigg, Lila Walker, Alan Wayne, G. A. Williams, Roscoe Woods, and the proposer.

Trigg pointed out that this problem was solved as problem 1459, *School Science and Mathematics*, vol. 36, p. 1031. It also appeared in the form $r=(abcd)^{1/2}/s$ in No. 472, *National Mathematics Magazine*, vol. 17, p. 182. For other properties of this quadrilateral see *School Science and Mathematics*, vol. 21, p. 280, problem 676 and vol. 31, p. 82, problem 1138. Woods located a trigonometrical solution of the problem in Durell and Robson, *Advanced Trigonometry*, (1930), pp. 27–28. It also appears in Shively, *An Introduction to Modern Geometry* (1939), p. 156, problem 26.

A proof of the famous Brahmagupta formula, $K=[(s-a)\ (s-b)\ (s-c)\ (s-d)]^{1/2}$, for the area of a cyclic quadrilateral may be found in many places. See, for example, Hobson, *Plane Trigonometry* (1928), p. 204 or Johnson, *Modern Geometry* (1929), p. 81.

Extreme Case of Lamé's Theorem

E 852 [1949, 104]. *Proposed by Roy Dubisch, Fresno State College*

A theorem due to Lamé states that the number of divisions D in the Euclidean Algorithm required to find the g.c.d. of two numbers a and b $(a>b)$ is never greater than $5p$, where p is the number of digits in b. While this result has been strengthened for special pairs a, b, show, by a counter-example with b as small as possible and corresponding a as small as possible, that the statement $D<5p$ for all a, b is false.

I. *Solution by Jacob Feldman, University of Pennsylvania.* Let a and b_0, $a>b_0$, be any two numbers such that $D=5p$, and let the D divisions of the Euclidean Algorithm for a and b_0 be

$$\begin{aligned} a &= b_0q_1 + b_1, \\ b_0 &= b_1q_2 + b_2, \\ &\cdots\cdots\cdots\cdots \\ b_{D-3} &= b_{D-2}q_{D-1} + b_{D-1}, \\ b_{D-2} &= b_{D-1}q_D. \end{aligned}$$

Let u_k be the kth term of the (truncated) Fibonacci sequence defined by $u_1=1$, $u_2=2$, $u_{n+2}=u_{n+1}+u_n$ for $n\geqq 1$. Notice that in the above set of divisions we have

$$b_{D-1}\geqq 1, \qquad b_{D-2}\geqq 2, \qquad b_{D-m}\geqq b_{D-m+1}+b_{D-m+2}.$$

Thus $b_{D-k}\geqq u_k$. Since $D=5p$, we have $D\geqq 5$ and

$$b_0\geqq u_5=8, \qquad b_1\geqq u_4=5, \qquad a\geqq b_0+b_1\geqq 13.$$

But for $a=13$, $b_0=8$, it is easily verified that $D=5p$. This, then, is the required counter-example.

II. *Solution by John Todd, Surrey, England.* The most slowly proceeding Euclidean Algorithms are those in which all the quotients are 1. Of these, that involving smallest numbers has its last remainder 1 and its penultimate remainder 2. Successive remainders in this case can be built up and are, in fact

$$(A) \qquad 1, 2, 3, 5, 8, 13, 21, 34, \cdots .$$

There are five divisions necessary to find the g.c.d. of 13 and 8. This is the counter-example required.

More generally, it is clear that the numbers (A) are those of the Fibonacci sequence, with the initial term $f_1=1$ omitted. This sequence is defined by $f_1=1$, $f_2=1$, and $f_{n+2}=f_{n+1}+f_n$ for $n \geqq 1$. It is known that

$$\sqrt{5}\, f_n = x^n - y^n, \qquad x = -1/y = (\sqrt{5}+1)/2 = 1.6180\cdots .$$

Further, the Euclidean Algorithm when applied to f_{n+2}, f_{n+1} involves n divisions. Now $p-1$ is the integral part of $\log_{10} f_{n+1}$, and this is asymptotically $(n+1)\log_{10} x$, since $|y|<1$. Hence

$$n \sim (1/\log_{10} x)p \doteq (4.785)p,$$

and certainly for sufficiently large n we have $n<5p$. Rough estimations show that this is true for $p \geqq 4$. Reference to a table of Fibonacci numbers shows that this is indeed the best possible result, for the Euclidean Algorithm, when applied to $f_{16}=987$ and $f_{17}=1597$, terminates after 15 divisions.

Also solved by Monte Dernham, Roger Lessard, C. D. Olds, and the proposer.

Lessard claimed that for $p \geqq 4$,

$$D < (4.785)p + 0.673,$$

which gives

$4 \leqq p < 8$	$D < 5p$
$8 \leqq p < 13$	$D < 5p - 1$
$13 \leqq p < 18$	$D < 5p - 2$
$18 \leqq p < 22$	$D < 5p - 3$
$22 \leqq p$	$D < 5p - 4$

It seems that $(a, b)=(13, 8), (144, 89), (1597, 987)$ are the only counter-examples.

Convergence of an Infinite Involution

E 853 [1949, 104]. *Proposed by C. S. Ogilvy, Trinity College*

If $y_1=x$, $y_2=x^{y_1}$, $\cdots$, $y_n=x^{y_{n-1}}$, what is the maximum x for which $\lim_{n\to\infty} y_n$ exists, and what is this limit?

Solution by P. G. Kirmser, University of Minnesota. If $\lim y_n$ exists, and is denoted by Y, then it is easily shown that $x = Y^{1/Y}$. It is also easy to show, by the standard method, that for $Y>0$, $x = Y^{1/Y}$ has an absolute maximum at $Y=e$. Now when $x=e^{1/e}$ we have $1<y_1<y_2< \cdots$ and, for all n, $y_n<e$, whence $\lim y_n$ exists. We readily find that $\lim y_n = e$.

Also solved by P. J. Burke, M. S. Klamkin, Joseph Rosenbaum, O. D. Smith, and the proposer.

John Todd pointed out that a solution to the problem is given on p. 38 of Francis and Littlewood, *Examples of Infinite Series with Solutions.* The problem is also found as ex. 23, p. 108 of Knopp, *Theory and Application of Infinite Series.* Here it is stated that the sequence $\{y_n\}$ converges if and only if $(1/e)^e \leqq x \leqq e^{1/e}$. Ex. 11, p. 23 of Bromwich, *Theory of Infinite Series* (2nd ed.), discusses a slightly more general problem than the proposed one by considering the convergence of the sequence defined by $v_n = a^{v_{n-1}}$, $v_1 = k$. Daniel Drew, a student at Reed College, used Bromwich's results to obtain the following theorem: "The sequence defined by $v_n = a_n^{v_{n-1}}$, where $a_n \to a$, $1 < a \leqq e^{1/e}$, converges to the smaller root of $x^{1/x} = a$ if there exists an r for which $v_r < P$, where P is the larger root of $x^{1/x} = a_p$ and $a_p = \underset{s \geqq r}{l.u.b.}\ a_s$. If $(1/e)^e \leqq a \leqq 1$, no conditions other than the convergence of $\{a_n\}$ are needed to ensure the convergence of $\{v_n\}$." The proof is direct by using *lim sup* and *lim inf*.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4360. *Proposed by Free Jamison, San Jose State College, California*

Any one of a group of airplanes may be refueled from any other. Each has a fuel capacity sufficient for a flight one-fifth the distance around the earth. Assuming that all have the same constant ground speed and the same rate of fuel consumption, that the only landing place and the only available fuel supply are at the home base, and that refueling time is negligible, find the minimum number of planes necessary so that one plane may fly around the earth and all return home safely. (Compare *The Jeep Problem* [1947, 24], and [1947, 458]).

4361. *Proposed by Melvin Dresher, The Rand Corporation*

If $S_1, S_2, \cdots, S_m$ are m line segments parallel to the y-axis such that through every set of $n+2$ of them the locus of an nth degree polynomial can be passed, then there exists some nth degree polynomial whose locus intersects all m segments.

4362. *Proposed by D. J. Newman, College of the City of New York*

Evaluate

$$\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{vmatrix}.$$

4363. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of positive upper density (*i.e.*, $\underline{\lim}\, a_k/k < \infty$). Then there exists an infinite subsequence such that no element divides another. In fact there exists an infinite subsequence $a_{i_1}, a_{i_2}, \cdots$ such that $\sum 1/a_{i_k} = \infty$ and no a_{i_k} divides any other.

4364. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

On the sides A_iA_{i+1} of an n-gon $A_1A_2 \cdots A_n$ as bases, isosceles triangles $A_iA_{i+1}B_i$ are constructed, either all exteriorly or all interiorly, with the vertex angle $B_i = 360°/n$. Prove

(a) If $A_1A_2 \cdots A_n$ is a projection of a regular n-gon, then $B_1B_2 \cdots B_n$ is regular.

(b) The problem of locating the points A_i when the points B_i are given is a porism.

SOLUTIONS

Monge Point of a Tetrahedron

4234 [1947, 112]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that if the Monge point of a tetrahedron $ABCD$ lies in the plane of the face BCD, then the altitudes through vertex A, of the triangles ABC, ACD, ADB are coplanar, and conversely. Examine the cases where the Monge point lies on an edge and at a vertex of the face BCD. Show how to construct tetrahedra illustrating each case.

Solution by René Blanchard, Le Havre, France.*

1. *Lemma.* In a tetrahedron:

(a). The distances of the Monge point from the faces are equal to one-half the segments intercepted on the altitudes by the planes of the faces and the circumsphere.

(b). The orthogonal projections of the Monge point on the faces are the midpoints of the line segments joining the orthocenters of the faces to the feet of the altitudes of the tetrahedron.

Let $ABCD$ be the given tetrahedron, (O) its circumsphere, G its centroid, Ω its Monge point, A' the foot of the altitude drawn from A, A'' the point where AA' meets (O) again, O_a' the midpoint of AA'' (orthogonal projection of O on AA'), (O_a), G_a, H_a the circumcircle, the centroid and the orthocenter of triangle BCD, G_a' and Ω_a the orthogonal projections of G and Ω on BCD.

(a). Ω is the symmetric of O with respect to G. We have

$$\vec{\Omega\Omega_a} = 2\vec{GG_a'} - \vec{OO_a}.$$

But $\vec{GG_a}=\vec{AA'}/4$ and $\vec{OO_a}=\vec{OA'}-\vec{AA''}/2-\vec{A'A''}$. Hence $\vec{\Omega\Omega_a}=\vec{AA'}/2-\vec{AA''}/2+\vec{A'A''}=\vec{A'A''}/2$.

(b) Since $(H_aG_a/H_aO_a)(\Omega_aO_a/\Omega_aG_a')(A'G_a'/A'G_a)=1$, H_a, Ω_a, A' are collinear. Furthermore, $H_a\Omega_a/H_aA'=(G_aG_a'/G_aA')(O_a\Omega_a/O_aG_a')=\frac{1}{2}$, and Ω_a is the midpoint of H_aA'.

2 (a). The necessary and sufficient condition that Ω be in the plane BCD is that $A'A''=0$, *i.e.*, that A' be on (O_a). Thus the projections of A' on the sides of the triangle BCD are collinear and the altitude of triangles ACD, ADB, ABC drawn from A are coplanar.

To construct such a tetrahedron it is sufficient to give the triangle BCD, to take an arbitrary point A' on (O_a), and to choose A as an arbitrary point on the perpendicular to the plane BCD erected at A'.

As A' described the circle (O_a), Ω describes the homothetic transform $(H_a, \frac{1}{2})$ of (O_a). The locus of Ω is the nine-point circle of BCD.

(b). If Ω is on CD, it can be either at the midpoint of CD or at the foot of the altitude of BCD from B on CD. To construct the tetrahedron it is sufficient to give the triangle BCD, to take the point A' symmetric to H_a with respect to Ω and to choose A arbitrary on the perpendicular to the plane BCD erected at A'.

(c). If Ω coincides with D, the altitudes of triangle BCD drawn from B and C meet at D. The angle at D of triangle BCD is a right angle, A' coincides with D, and the tetrahedron is trirectangular at D.

A Diophantine System

4284 [1948, 100]. *Proposed by E. T. Bell, California Institute of Technology*

"Diophantus [II, 20 and IV, 45] proposed to find three squares such that

* Translated by W. E. Byrne, Virginia Military Institute.

$$y^2 - x^2 : z^2 - y^2 = a : b,$$

where $a:b$ is a given ratio." (L. E. Dickson, *History of the Theory of Numbers*, v. 2, p. 419.) For each of $a/b = 3, 1/3$ he gave one rational solution. Prove that the complete integer solution is given by

$$2x = k(a_1b_1g^2 + 2agh - a_2b_2h^2),$$
$$2y = k(a_1b_1g^2 + a_2b_2h^2),$$
$$2z = k(a_1b_1g^2 - 2bgh - a_2b_2h^2),$$

where a_1, a_2 are any integers whose product is a, and b_1, b_2 are any integers whose product is b, and k (for any pair of integer values of g, h) is an arbitrary integer multiple of the reciprocal of the G.C.D. of b_1g+a_2h, a_1g-b_2h, $a+b$.

Solution by Roger Lessard, École Polytechnique, Montreal. We have

$$\frac{a}{b} = \frac{(x-y)(x+y)}{(y-z)(y+z)}. \tag{1}$$

Then there exist integers g, g', h, h', such that

$$x + y = gh'a_1, \qquad x - y = g'ha_2, \tag{2}$$

$$y + z = gg'b_1, \qquad y - z = hh'b_2, \tag{3}$$

with $a_1a_2 = a$, $b_1b_2 = b$. These give

$$2y = gh'a_1 - g'ha_2 = gg'b_1 + hh'b_2,$$

whence

$$g'/h' = (ga_1 - hb_2)/(gb_1 + ha_2). \tag{4}$$

Using (4) it is easy to reduce (2) and (3) to the form $x:y:z = x':y':z'$, where

$$x' = a_1b_1g^2 + 2agh - a_2b_2h^2,$$
$$y' = a_1b_1g^2 + a_2b_2h^2, \qquad z' = a_1b_1g^2 - 2bgh - a_2b_2h^2.$$

Editorial Note. If d is the G. C. D. of x', y', z', we have for some integer s,

$$x = sx'/d, \qquad y = sy'/d, \qquad z = sz'/d. \tag{5}$$

Conversely, if g, h, s are arbitrary integers, then x, y, z as given by (5) will satisfy (1), as may be verified by direct substitution.

The attempt to be more specific than (5) leads to some difficulty. Let f be the G.C.D. cited by the Proposer and m the arbitrary integer multiple, so that $k = m/f$. The Proposer's statement is equivalent to (5) if and only if $d = 2f$. We may write

$$x' = a_2hu + a_1gv, \qquad z' = b_1gu - b_2hv, \qquad y' = -a_2hu + a_1gv = b_1gu + b_2hv,$$

where $u = a_1g - b_2h$ and $v = b_1g + a_2h$. It is thus evident that f divides d. (It is easy to show that the G.C.D. of u and v is always a divisor of $a+b$.)

Now unless a_1, a_2, b_1, b_2 are chosen in a special manner, d will not equal $2f$ and the solution in terms of k may be incomplete or may yield fractional values for x, y, z. For example, if $x=1$, $y=19$, $z=29$, we have $a=3$, $b=4$. Of the six ways of selecting the factors, only one way leads to $d=2f$. The others give $d=12f$, $4f$, $3f$, $3f$, f. Thus the Proposer's formulas fail to give the solution 1, 19, 29 at all in four of these cases, while in three of them the solution 3/2, 57/2, 87/2 appears.

Film Wound on a Reel

4285 [1948, 165]. *Proposed by H. N. Davis, Victor Erikson, and Robert Nathans, United States Army*

A full reel of film (thickness τ) of original diameter A is being wound onto an empty spool of original radius a and rotating at a uniform angular velocity ω. How long does it take to unwind the first spool if its inner diameter is also a?

Solution by W. B. Campbell, Philadelphia Textile Institute. If the film being wound upon the spool is assumed to take the form of a spiral

$$r = a + (\theta/2\pi)\tau^*$$

and if $d\theta/dt=\omega=$constant, we have $dr/dt=\tau\omega/2\pi$. The time for attaining a specific value of r is $\int_a^r(dt/dr)dr$. Since the final value of r is A, the same as on the first spool, the total time is $2\pi(A-a)/\tau\omega$. As a check, the approximate number of turns is $n=(A-a)/\tau$, and the number of revolutions per unit of time is $R=\omega/2\pi$, making the total time $=n/R$. This solution neglects the (unstated) distance between axes.

A matter of possible interest is the length of the film. An obvious estimate for it is obtained from $n=(A-a)/\tau$ and the estimated mean radius $R_m=(A+a)/2$, so that the length $L=2\pi R_m n=\pi(A^2-a^2)/\tau$. The full expression is

$$L = \int [dr^2 + (rd\theta)^2]^{1/2} = \int_a^A [1 + (2\pi r/t)^2]^{1/2} dr.$$

This responds readily to formal integration, but it is interesting to note that if the first term under the radical is suppressed, as negligible in comparison with the second, the result is identical with the estimated value of L.†

The spiral law for the form of the film seems as reasonable as any, since we do not know the shape of the end of the film, nor how it is fastened onto the spool, nor what the winding tension is, nor the nature of the elastic deformation by which the influence of the "hump" becomes negligible as turns accumulate upon the coil.

* We make the simplifying assumption that a and A in the proposal are meant to represent radii on both spools, a reasonable reading in light of the word "also."

† Assuming the spiral form, the length of one turn at radius r_i, expanded in powers of τ, is $L_i=2\pi r_i+\pi\tau+\cdots$. Hence neglect of the first term under the radical gives the same result as assuming $L_i=2\pi r_i$, which is also equivalent to the assumption of circular layers made by some of the other solvers.

Also solved by F. W. Herlihy, C. S. Ogilvy, Mary H. Payne, Hanan Rubin, and G. W. Walker.

Editorial Note. The Proposers were interested in the acceleration of the first spool. Suppose r', r are the radii of the outer layers on the two spools at time t. Then the lengths of film on the spools are $\pi(r'^2-a^2)/\tau$, $\pi(r^2-a^2)/\tau$, approximately, as in the above solution. But the total length is $\pi(A^2-a^2)/\tau$. Hence $r'^2+r^2=A^2+a^2$. By differentiation and substitution of $dr/dt=\tau\omega/2\pi$ we find for the angular velocity of the first spool $\omega'=r\omega/r'$, and for its angular acceleration

$$d\omega'/dt = \omega^2\tau(A^2 + a^2)/2\pi r'^3.$$

Representation of Integers in the Form: a kth Power Plus a Prime

4287 [1948, 165]. *Proposed by C. R. Phelps, Rutgers University*

Show that for any given integer $k>1$, there are an infinite number of perfect kth powers which cannot be written as the sum of a prime and a kth power. (This disproves a conjecture of Hardy and Wright, *Introduction to the Theory of Numbers*, p. 19.)

Solution by P. A. Clement, University of California at Los Angeles.

(1) The function $(x+a)^k-x^k$, $k>1$, is algebraically factorable and thus can represent a prime for integral x only if $a=1$.

(2) Let $F(x)=(x+1)^k-x^k$ and, for an arbitrary positive integer x_0, let $q=F(x_0)$. Then q is an integer >1. Putting $y_n=x_0+nq$, $n=1, 2, \cdots$, we have

$$\begin{aligned} F(y_n) &= (y_n + 1)^k - y_n^k = [(x_0 + 1 + nq)^k - (x_0 + nq)^k] \\ &\equiv (x_0 + 1)^k - x_0^k = q \pmod q. \end{aligned}$$

Hence, for all n, $F(y_n)$ is divisible by q, but is not equal to q since $F(x)$ is obviously an increasing function of its argument. Thus, for each choice of x_0, $(y_n+1)^k$ gives an infinite set as required.

Also solved by Robert Breusch, Paul Erdös, William Gustin, W. J. Harrington, Fritz Herzog, Free Jamison, Roger Lessard, Leo Moser, E. Trost, and the Proposer.

Editorial Note. Breusch, Harrington, and Moser cited the set A^k, where $A=n(2^k-1)+2$, $n=1, 2, \cdots$. Several solvers employed the known theorem that any polynomial $F(x)$ with integral coefficients, not a constant, is composite for infinitely many values of x (Hardy and Wright, p. 18). An obvious slight change in the above proof will establish the general theorem.

The Proposer noted that the case $k=2$ was treated by Lischinsky and Webber, *Transactions of the Royal Society of Canada*, v. 27, pp. 71–90 in which Hardy-Littlewood methods are used to show that "almost all" numbers can be represented in the form y^2+p. The Proposer also suggests, in place of the conjecture which lead to the problem, the following: Corresponding to each sufficiently large N there exist integers y, $k(>1)$, and p (prime) such that $N=y^k+p$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Unified Calculus. By E. S. Smith, Meyer Salkover, and H. K. Justice. New York, John Wiley and Sons, Inc., 1947. 10+534 pages. $3.50.

The text represents an excellent way to present to the student as early as possible the fundamental notions of differential and integral calculus. After an especially good ten-page chapter on the limit idea the authors introduce differentiation. This is well motivated by the idea of speed and is one of the best presentations the reviewer has seen. With just the necessary formulas for differentiating a polynomial the authors discuss curve tracing as applied to polynomials. Next follows a section on applications of maxima and minima with a very good set of story problems.

After other applications of derivatives the notion of integration is introduced with Chapter 3. Integration as inverse of differentiation and as the limit of a sum is applied to find areas, solids of revolution, *etc.* Thus in the first 73 pages the beginning student is given an excellent introduction to the whole elementary course.

Next comes a chapter on centroid, moment of inertia, work, *etc.* It is not until Chapters 5 and 6 that differentiation formulas for algebraic and transcendental functions are derived. As the differentiation of each new type of function is introduced a list of problems on applications is given. This keeps the student systematically reviewed on maxima, minima, inflection points, *etc.* After two other chapters on applications of derivatives, Chapters 9–12 complete the usual elementary work on integration. The remaining chapters are devoted to the customary topics of series, expansions of fractions, hyperbolic functions, partial differentiation, multiple integrals, and some differential equations. In the back are formulas from previous elementary courses and tables of integrals, trigonometric functions with the argument in degrees and radians, natural logarithms, hyperbolic functions, and powers of e.

The reviewer feels that the definitions of maxima, minima, and inflection points on a curve are poorly done. They lead one to believe these points can occur only when the function and its first two derivatives are continuous.

Otherwise the book seems very good. There is an ample number of well graded exercises in each set to use the book two years without repeating. Answers to the odd numbered exercises are given.

All illustrative examples and exercises are in a slightly smaller type than the body of the text, but the type is clear and the spacing good. The text makes a very pleasing appearance. The reviewer did not find a single typographical error!

An outstanding feature of the book is the set of several hundred diagrams. Those showing the element of integration in solids are especially well done.

J. A. WARD

Mathematics: Our Great Heritage. Essays on the Nature and Cultural Significance of Mathematics. Edited by W. L. Schaaf. New York, Harper and Brothers, 1948. 11+291 pages. $3.50.

"These essays are *about* mathematics. They have been selected for the thoughtful reader who would understand why mathematics means so much to mankind. There is little in them that is technical—not, at least, in the sense that they bristle with esoteric symbols and intricate diagrams" (from the preface). Culled from the expository writings of well-known figures in the mathematical world, the essays focus on five aspects of mathematics:

I. *The Creative Spirit.* "Mathematics as an Art," by J. W. N. Sullivan; "On the Seriousness of Mathematics," by G. H. Hardy; "Mathematics—the Subtle Fine Art," by J. B. Shaw.

II. *Wellsprings.* "On the Development of Mathematics," by E. T. Bell; "On the Genesis of Mathematical Ideas," by George Sarton; "On the Sociology of Mathematics," by D. J. Struik.

III. *The Queen.* "An Introduction to Modern Mathematical Thought," by C. V. Newsom; "On the Nature of Mathematical Truth," by C. G. Hempel; "The Two Realities," by Tobias Dantzig.

IV. *The Handmaiden.* "Mathematics and the Sciences," by Tomlinson Fort; "Mathematics and the Sciences," by J. W. Lasley, Jr.; "On the Relation of Mathematics and Physics," by R. B. Lindsay; "Industrial Mathematics," by T. C. Fry.

V. *Humanistic Bearings.* "On the Nature of Mathematical Knowledge," a report of the Progressive Education Association; "Mathematics and the Humanities," by Archibald Henderson; "Mathematics as a Cultural Bridge," by Arnold Dresden; "The Larger Human Worth of Mathematics," by R. D. Carmichael.

That such a collection of essays should make for splendid reading goes without saying. And they do. The professional mathematician will single out many stimuli for his own ideas, many foci for further development of thought and exposition on his part. Here are said a large number of the things he has always intended to say, often better said than he could say them; the reviewer even finds (to his amusement and chagrin) the likening of mathematics to a juice extractor which he has always considered his very own simile. The student of mathematics, graduate student or undergraduate major, will achieve an orientation and synthesis of hitherto compartmentalized knowledge, a clearing of trees and stumps into greener pastures. He will derive comfort and peace of mind from Carmichael's inspiring summary of the effect of a life-time devotion to mathematics on that life. The beginner in mathematical specialization who won-

ders what to do with his specialty "if he does not want to teach it," will profit from the directive contained in Fry's guide to industrial mathematics and in the essays on the counterpoint of mathematics and the sciences. As collateral reading material the collection will serve as a survey course in general mathematics with its store of topics ripe for further investigation by the thoughtful student or for further amplification by the instructor. As for an intellectually curious lay reader, one with even a modicum of mathematical background should have his eyes opened to many of the meanings and beauties which have escaped him before. And there are many such readers, still insatiably and unsuccessfully questing for knowledge of a field of thought which only their intuition tells them is important.

Every reader will have personalized preferences for some of the essays or parts of essays. Indeed, personal preference on the part of the editor was one of the guides to selection. And any reader, himself his own reviewer, may quarrel with the omission of some favorite essay and the inclusion of another which does not quite catch his fancy. It is therefore not as an index of what the reader should extract from the book that the present reviewer claims the privilege of recording his own preferences and complaints. He is grateful indeed for the beauty of Carmichael's tribute to the human worth of mathematics; for the disarming directness of Hardy's contrast of what is serious in mathematics, because it is forward looking and developable, with what, though fine and interesting, is yet dead end; for the lucidity of Hempel's exposition of the postulational structure of mathematics. He must admire the timely brilliance of Dresden's advocacy of achieving "necessary and sufficient conditions" for an "existence theorem" for world peace, and of securing better intercultural relations by an application of the study of "inverse functions"—"putting one's self in the other man's shoes," the *man muss immer umkehren* of Jacobi in social dress. But he must also confess that, although he ardently contends that mathematics is indeed a subtle fine art, he does not find the artificially artful arguments of Shaw on the subject persuasive to the contention.

Acknowledging the difficulties of selection confronting the editor and his general skill in solving them, the reviewer feels that in some respects the selections were not wisely balanced. Why *two* essays on mathematics and the sciences? Are not the essays of Newsom and Hempel largely co-extensive for this project? Indeed, omission of one of each of these pairs would have opened up space for the further development of Part III, the least convincing section of the book to this reviewer. For, the queenliness of mathematics is attested not alone by its logical formulation and its philosophy of the infinite. What of the unifying concept of function and the invention of techniques for the extraction of information from a functional relationship? What of the central rôle of form and transformation and invariant in algebra and geometry? Is it not a royal thing that a cache of mathematics has been, as mathematics for mathematics' sake, conceived and developed, and stored ready to the hand of the grateful scientist

of a later period? These themes can surely be made vivid to the general reader; and surely there exist articles discussing them which are non-technical and do not "bristle with esoteric symbols." If such are indeed not available (in English or in translation from foreign languages), then it might have been wise of the editor to have commissioned their writing, for without them the beautiful all-pervasiveness of mathematics is not brought home to the reader.

Despite these relatively minor criticisms, the general verdict of this review is that the book is admirable, successfully sustaining a high level of content in pursuit of a worth while purpose.

G. M. MERRIMAN

Differential- und Integralrechnung im Hinblick auf ihre Anwendungen. By Louis Locher-Ernst. Basel, Verlag Birkhäuser, 1948. 594 pages. Fr. 48.

This uncommonly rich text is devoted principally to plane analytic geometry and differential and integral calculus of functions of a single real variable. Although it is the author's aim, on the basis of extended teaching experience, to develop his material ab initio, with emphasis on its applications, he gives relatively few specialized applications. Instead, he has attempted to choose content appropriate for application and to present it in a form technically adapted to application. He anticipates that the "abstract" mathematician may find too much numerical work, and that the technician may find too much genuine mathematics.

The most prominent single feature of the book is the comparatively extensive treatment of numerical and graphical methods. Difference tables are introduced immediately after the calculus of polynomials, and the interpolation formulas of Newton, Bessel, and Stirling are obtained. In the introductory discussion of the definite integral, approximations by the trapezoidal rule and Simpson's rule follow integration of the linear function. Numerical and graphical differentiation, integration, and harmonic analysis are discussed rather fully. The treatment of Taylor series and the expansions of the elementary functions (including the binomial) are based in each case on a careful discussion of the remainder. In practically all numerical work, exact hypotheses and a precise estimate of error are demanded.

Over a fifth (103 pages) of the text is devoted to the conic sections, differential geometry of plane curves, rollers, and kinematic geometry. On the other hand, one finds no general discussion of infinite series of constants (except the geometric series), and the convergence interval of a power series is defined only by uncertain implication (p. 431, italics). Although, in fact, the interval is invariably indicated by an appropriate inequality, the reviewer encountered no mention of the convergence radius.

In accordance with accepted pedagogical practice, the author approaches a fundamental idea first from a descriptive or intuitive point of view, cites illus-

trative examples and results, then gives a rigorous definition, and proceeds to develop, extend, and apply the concept. It seems open to question, however, whether any good purpose is served by an introductory remark on the definite integral, (p. 130) stating that "Das Product $f(x)dx$ bedeutet den Inhalt eines unbegrenzt schmalen rechtecks. . . . " Again, the introductory statement (p. 166) on nullsequences says that the " . . . Glieder mit unbegrenzt wachsender Nummer sich immer weniger von Null unterscheiden . . . , " seeming to imply monotonicity; a rigorous definition follows a few lines later. One reads also (p. 164) that the equation $\lim x = x_1$, " . . . soll aber nicht den sinn haben, dass x die Grenze x_1, unbedingt erreichen muss, sondern nur, dass x ihr jedenfalls beliebig nahe kommt." Although cluster points are not used in this text, even a first view of the limiting operation should distinguish unambiguously between convergence and nonconvergence.

Perhaps the most questionable passages refer to differentials and related notions. A differential is defined (p. 163) as follows: "Eine variable grösse, die unbegrenzt dem Werte Null zustrebt, nennen wir eine werdende Null oder ein Differential." Increments Δx and Δy are defined in the usual way and are used in the usual way in treating the derivative of a polynomial. In connection with a more general treatment of differentiation, Δx and Δy are replaced (p. 185) *when approaching zero* by differentials: "$dx = \Delta x \to 0$, $dy = \Delta y \to 0$." The limit of the difference quotient (p. 174) is "kurz dv/du; genauer wäre die Screibewise lim (dv/du)." Finally (p. 191) we learn that $dy = f'(x)dx$ is not an equation in the ordinary sense, but that it must be understood as a "Grenzwertgleichung"; the variable functional increment dy has to the corresponding increment dx a ratio whose limit is $f'(x)$. Comment is offered (pp. 191–192) on the more conventional view, in which, for fixed x, dy and dx are two variables whose ratio is $f'(x)$, and $dy = f'(x)dx$ is the equation of the tangent line at (x, y) in running coördinates (dx, dy) relative to (x, y).

It is stated explicitly (p. 166) that $+\infty = -\infty$, meaning that there is only one special number ∞, so that $\pm\infty$ is more properly $\infty\pm$. One is led to admit (p. 165) that, if $du \to 0\pm$ then $1/du \to \infty\pm$. (Why not $\infty \mp$?)

There are several other less important departures from conventional notation and terminology. For example, for the point P with coördinates (x, y) the notation Px/y is used far more often than the usual $P(x, y)$. Again, the "main theorem of differential and integral calculus" is (p. 251) that, if $|f'(x)| \leqq M$ for $a \leqq x \leqq b$, then $|f(x) - f(a)| \leqq (x-a)M$ for $a \leqq x \leqq b$.

It seems inconsistent with the otherwise highly articulate character of the book that only intuitive evidence is offered (p. 324) for the existence, under suitable conditions, of a solution $y = F(x)$ to the equation $f(x, y) = 0$.

A section of twenty-five pages is devoted to solid analytic geometry and the differential calculus of functions of several variables. There are 53 short biographical sketches, brief tables, over 400 attractive diagrams, and over 1000 problems. One also finds much intuitive and motivational material and other

detail normally reserved for the classroom. Though the print is small and displays are compact, the book is easily read and style and format are quite attractive.

It is the reviewer's opinion that, on the whole, the advantages of this elaborate text outweigh its disadvantages. The latter can be overcome by effective instruction. Aside from the wealth of exercises and much attractive expository detail, the principal advantage consists in a presentation of graphical and numerical methods that is thoroughly integrated with the rest of the text; such a presentation appears not to have been available previously on an elementary level. In any case, this will be a very useful reference book for those teaching the subjects it covers.

F. A. FICKEN

NEW BOOKS RECEIVED

College Algebra. Revised Edition. By L. M. Reagan, E. R. Ott, and D. T. Sigley. New York, Rinehart, 1948. 16+447 pp. $4.00.

An Introduction to College Geometry. By E. H. Taylor and G. C. Bartoo. New York, Macmillan, 1949. 8+143 pp. $3.15.

Higher Algebra for the Undergraduate. By M. J. Weiss. New York, Wiley, 1949. 8+165 pp. $3.75.

Probability Theory for Statistical Methods. By F. N. David. Cambridge, University Press, 1949. 10+230 pp. $3.50.

The Mathematics of Circuit Analysis. By E. A. Guillemin. New York, Wiley, 1949. 14+590 pp. $7.50.

Contributions to Applied Mechanics. By H. J. Reissner (Anniversary volume). Ann Arbor, J. W. Edwards, 1949. 8+493 pp. $6.50.

Theory and Application of $\int_0^z e^{-z^2}dx$ *and* $\int_0^z e^{-p^2y^2}dy \int_0^y e^{z^2}dx$. (Part I, Methods of Computation.) By J. B. Rosser. New York, Mapleton House, 1948. 4+192 pp. $8.00.

Mathematics Review Exercises. Revised Edition. By D. P. Smith and L. T. Fagan. New York, Ginn, 1949. 8+280 pp. $2.00.

Modern-school Solid Geometry. New Edition. By R. R. Smith and J. R. Clark. Yonkers-on-Hudson, World Book, 1949. 8+256 pp. $1.76.

Living Mathematics. 2nd Edition. By R. S. Underwood and F. W. Sparks. New York, McGraw-Hill, 1949. 10+374 pp. $3.00.

Sampling Methods for Censuses and Surveys. By F. Yates. London, Griffin, 1949. 14+318 pp. 24s.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Sigma Mu Pi, Honorary Mathematics Fraternity

Sigma Mu Pi had its origin in 1947 at Asheville-Biltmore College, Asheville, N. C. Its principal activity is to recognize outstanding mathematical scholarship among college students. A miniature slide rule has been adopted as the fraternity key. Further information concerning the fraternity may be obtained by writing to Mr. Thomas Case of Asheville-Biltmore College, the National Secretary.

CLUB REPORTS, 1948–49

Pi Mu Epsilon, St. Louis University

The papers heard by the St. Louis University chapter of *Pi Mu Epsilon* were:

Generating functions, by Rev. Charles Rust, S.J.

Newton's polygon in higher plane curves, by Rev. J. E. Case, S.J.

Special methods of integration, by Charles Koerner

The concept of area in plane geometry, by Prof. C. C. MacDuffee, University of Wisconsin and Director General of Pi Mu Epsilon

The prize winners in the third annual Essay Contest were: Jeanette Maschmann, for her paper *Carl Frederick Gauss, His Life and Work;* Sister Ermelinda Van Domelen, of the Sisters of St. Mary, for the paper *Gauss' fundamental theorem of algebra.*

The officers elected during the year were: Director, Edward Thirkhill; Vice-Director, William Felling; Secretary, Virginia Herre; Faculty Adviser, Dr. Francis Regan.

Mathematics Club, Wellesley College

The Wellesley *Mathematics Club* enjoyed the following talks during 1948–49:

Chinese numerals and their use, by Miss Zung-Nyi Loh

Educational experiences in Vienna and Great Britain, by Miss Ilse Novak

Teaching of algebra and geometry in secondary schools, by Jasper Moulton, Mathematics Department of Wellesley High School

Modern algebra, by Joyce Friedman

Differential equations, by Mary Ann Berry

Boolean algebra, by C. Elizabeth Taylor.

Club members attended the Wellesley Science Conference as part of their activities.

The officers for the year 1948–49 were: President, Carol Rogers; Vice-President, Joyce Friedman; Treasurer, Lindsley Clark; Secretary, Diane Gruhler; Junior Executive, Betsy Martin; Sophomore Executive, Rachel Allen; Faculty Advisor, Miss Marion Stark.

The officers for 1949–50 are: President, Diane Gruhler; Vice-President, Florence Van Dyke; Secretary, Elizabeth Robinson; Treasurer, Elizabeth Weiner; Junior Executive Ursula Loengard; Faculty Advisor, Miss Ilse Novak.

Kappa Mu Epsilon, Pittsburg (Kansas) State Teachers College

The *Kansas Alpha* chapter of *Kappa Mu Epsilon* presented the following topics at six open meetings:

Origins of mathematical symbols, by Prof. J. A. G. Shirk
The scope of mathematics, by Dr. R. G. Smith
Addition-subtraction logarithms, by Prof. F. C. German
The number 2 in mathematics, by James Hudson
Career fields for mathematicians, by Norval Phillips
Mathematics in secondary schools, by Miss Jane Townsend, principal of the Girard High School
Vital statistics, by D. E. Waggoner, Director of the Division of Vital Statistics, Topeka, Kansas.

Joint meetings were held with the Biology Club and with the Physical Science Club. Kappa Mu Epsilon keys were awarded to James Hudson and Frank Slane for the highest scholastic standings in mathematics and general scholarship.

The officers for the past year were: President, James McCollam; Vice-President, James Hudson; Secretary, Betty Multhaup; Treasurer, Norval Phillips; Corresponding Secretary, Prof. J. A. G. Shirk; Sponsor, Dr. R. G. Smith.

Mathematics Club, University of Kansas

The *Mathematics Club* of the University of Kansas presented the following programs during 1947–48:

Careers in mathematics, by Dr. G. B. Price
Thales, by Arnold Wedel
Number systems, by John Michener
Probability, by Bertha Cummins
Cryptography, by Dr. G. W. Smith
Sine curves, by Joseph Hull
Meteorology, by Vernon Benson
Doubling the cube, by Dr. Robert Schatten
Hessian line co-ordinates, by Francis Brooks
Pythagorean triples, by Charles Terry
The boxing-in process, by W. K. Moore

Mr. Francis Brooks was awarded the book *Men of Mathematics* for presenting the best undergraduate talk of the year. The annual picnic was held in May.

Officers elected for the year 1948–49 are: President, Charles M. Terry; Vice-President, Elneta Richmond; Secretary-Treasurer, Christine Mann.

Kappa Mu Epsilon, Mount Mary College

The *Wisconsin Alpha* Chapter of *Kappa Mu Epsilon* held eight regular meetings during 1947–48. The main business of the year was the drawing up of chapter by-laws and the reading of short papers by the members of the chapter. The following papers were read:

Symposium on non-Euclidean geometry, by Eileen Ford, Pat Farrell, Eleanor Grogan, Mildred Oestreich, Mary Lou Marquardt, Janet Kuhn, and June Rose McDonald

Napier, by Audrey Reiff

Logarithms, by Dorothy Karner

Infinity, by Norma Harding.

The social events for the year included a card party and a dinner for new initiates.

Officers for 1948–49 are: President, Norma Harding; Vice-President, Bernadine Spitznogle; Secretary, Mary Alice Gauerke; Treasurer, Marilyn Briggeman; Corresponding Secretary and Faculty Sponsor, Sister Mary Felice.

Pi Mu Epsilon, Michigan State College

Participation in presentation of mathematical topics by both undergraduates and faculty members was the objective of the *Michigan Alpha* Chapter of *Pi Mu Epsilon* during 1947–48. The following programs were offered:

Mathematical problems in geography, by Joyce Clark

An envelope of pedal lines of a triangle, by Dr. B. M. Stewart

A two variable maximum problem, by Joyce Deisch and Eugene Parker

Using IBM cards to find prime numbers, by Dr. J. S. Frame

Radar equations, by Dr. J. H. Bell

A statistics program, by Robert Zavell and Dale Hekhuis

Calculus of variations or history of mathematics, by Dr. J. E. Powell

Is mathematics important?, by Prof. C. C. MacDuffee of the University of Wisconsin and Director General of *Pi Mu Epsilon*.

The L. C. Plant awards based on scholarship, interest in mathematics, and helpfulness to the mathematics department were presented to James Powell (first prize of $50.00) and to Richard Zindler (second prize of $40.00). Fifty-nine new members were admitted during the year.

Officers elected for 1948–49 were: President, James Powell; Vice-President, Robert Houston; Secretary, Dale Hekhuis; Treasurer, Wendell Grove; Faculty Advisors, Dr. J. H. Bell and Mrs. B. B. Houston.

Pi Mu Epsilon, Michigan State College

The *Michigan Alpha* Chapter of *Pi Mu Epsilon* held ten meetings during the year 1948–49 with mathematical papers being presented by both faculty and student members. Picnics were held during the spring and fall terms and the annual banquet was held during the winter term. At the banquet, which was attended by sixty-six members and guests, Dr. R. V. Churchill of the University of Michigan spoke on *Some applications of differential equations*. Two formal initiations were conducted at which the sixty-four initiates gave short biographical talks on famous men in mathematics.

The following papers were presented at the regular meetings:

Calculus of variations, by Dr. J. E. Powell

Solutions of oblique triangles without tables, by Dr. H. E. Stelson

Three solutions of a mathematical problem from the standpoint of calculus, theory of numbers, and probability, by James Collins, Alex Arnot, and Robert Houston

Theory of knots, by Dr. E. A. Nordhaus

What is infinity?, by Philip Hartman and Edward Seligman

Officers for the year 1948–49 were: President, James Powell; Vice-President, Robert Houston; Secretary, Dale Hekhuis; Treasurer, Wendell Grove; Faculty Advisors, Dr. James Bell and Barbara Houston.

Mathematics Club, Boston University

At the first meeting of the *Mathematics Club of* Boston University for the year, Professors Mode, Johanson, Sobczyk, and Syer spoke on *Occupational opportunities in Mathematics*. Other speakers at the bimonthly meetings were:

What are your chances?, by Mr. Arvanitis

Curved and flat spaces, by Prof. Sobczyk

"*Whom the Gods Love*," by Miss Nickerson

Peculiar topological curves, by Dr. Giever

Significant figures, by Mr. Stubbs

Three famous Greek problems, by Mr. Olds

Solutions with solutions, by Mr. Whitcomb

Magic squares, by Mr. Twigg.

Representatives of the club attended two meetings of the Greater Boston Intercollegiate Mathematics Clubs Association. The Harvard Club was host at the first meeting at which Mr. A. M. Gleason of Harvard spoke on *Nim and oriented graphs*. The second assembly was held at Massachusetts Institute of Technology where Prof. Dirk Struik spoke on *Zeno's paradoxes*.

The social activities included a Halloween Party, a supper, a bowling party, a self-prepared Spaghetti Supper and Gym Night, a Theatre Party, and a picnic.

Officers elected for 1949–50 are: President, Irene Calnan; Vice-President, John Swaffield; Secretary, Marjorie Radcliffe; Treasurer, Bernard Olshansky.

Kappa Mu Epsilon, Iowa State Teachers College

The *Iowa Alpha* Chapter of *Kappa Mu Epsilon* at Cedar Falls, Iowa held an Alumni Homecoming Breakfast and two formal initiation dinners in addition to the regular meetings at which the following talks were presented:

Non-cummutative processes, by Loren Sheldahl
Some tricks of paper cutting, by Donna Whiting and Betty Sayre
Postulates of ordinary geometry, by Jim McGrew
Mathematical induction, by Bill Boettcher
When you fly, by Robert Lankton
Finding extremes by algebraic means, by David McClure
Congruencies, by Richmond Trunkey
An alignment chart for the quadratic equation, by Mrs. Robbie Lou Ashworth.

The *Iowa Alpha* Chapter was well represented at the National Convention of *Kappa Mu Epsilon* held at Topeka, Kansas, April 10–12. Eight students and two faculty members attended the sessions at which Dr. H. Van Engen was re-elected National President.

The officers for 1948–49 were: President, George Mach; Vice-President, Orval Knee; Secretary-Treasurer, Lena Abbas; Corresponding Secretary, George Keppers.

Pi Mu Epsilon, Louisiana State University

Louisiana Alpha Chapter of *Pi Mu Epsilon* held its first meeting of the year for the purpose of organizing the calendar for the session. Papers presented at the regular meetings included:

Applications of complex numbers, by Prof. B. B. Townsend
Coördinate systems, by M. C. Wicht
Mathematics of the atomic bomb, by Prof. F. B. Rickey
Straight line construction of conic sections, by Ernest Ikenberry
LaPlace transforms, by Miss Margaret LaSalle
Osculating figures, by Donald Shipp.

The *Pi Mu Epsilon* Lecture Series sponsored by the local chapter was delivered by Prof. Saunders MacLane of the University of Chicago. His topics included *What is topology?* and *Co-homology theory for groups.*

The year was completed with the initiation of sixty-one members. Following the initiation, a banquet was held at which the two annual awards were presented. For making the highest grade on the freshman honors mathematics examination, James Turner, Jr. was recognized; Mrs. Christine Whitman and Marion Smith were honored jointly as outstanding Seniors in mathematics.

Officers for the year 1948–49 were: President, Donald Shipp; Vice-President, Marion Smith; Secretary, Alice Pecot; Treasurer, Frank Woolam; Faculty Advisor and Corresponding Secretary, Prof. H. T. Karnes.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

WAYNE UNIVERSITY ACQUIRES A MATHEMATICAL MACHINE

A complex mathematical machine, designed and built in 1931 by Dr. Vannevar Bush, became the property of Wayne University in September as the result of a gift from the Massachusetts Institute of Technology. The machine is in two units known as differential analyser and cinema integraph. The machine will find the solution to differential equations of considerable complexity. As a result of this gift, the University can be of greater service to local industry in their solution of research problems.

SCIENTIFIC JUBILEE OF PROFESSOR MAURICE FRÉCHET

The following announcement concerning the "Jubilé Scientifique Du Professeur Maurice Fréchet" has been received by Professor T. H. Hildebrandt and is herewith in translation brought to the attention of the members of the Association:

Professor Maurice Fréchet is approaching the age of retirement. His friends colleagues, and pupils desire to express to him on this occasion, their admiration, their affection, and their gratitude by celebrating his Scientific Jubilee.

Professor Fréchet has expressed the wish that this occasion should be characterized by the utmost simplicity. Near the beginning of the year 1950 one or two prizes will be awarded to an author, or authors, of a memoir on general analysis. The prize winners will be selected by a committee appointed by the Council of the Société Mathematique de France and their names will be announced at one of the sessions of this Société. The Bulletin de la Société will publish later the name or names of the prize winners, as well as a list of subscribers.

In keeping with the expressed wish of Maurice Fréchet this letter is signed only by the members of the section in geometry of the Academie des Sciences, the members of the Council of the Société Mathematique and colleagues and former colleagues of Maurice Fréchet on the Faculty of Sciences of the Collège de France and l'École Polytechnique.

E. Borel, E. Cartan, Denjoy, Hadamard, Julia, Montel.

Brard, president; Bayard, Belgodere, Benoit, Boos, Cagnac, H. Cartan, Chatelet, Choquet, Courbon, Courtand, Desforge, Mme Dubreil, Fourès, Janet, Jean, Lamothe, LeLong, Leray, Lichnerowicz, Maillard, Mandelbrojt, Marchand, Schwartz.

Beghin, Borel, Bouligand, Brard, de Broglie, E. Cartan, H. Cartan, Chapelon, Chatelet, Chazy, C. Darmois, Denjoy, Dubreil, Favard, Garnier, Hadamard, Janet, Julia, Leray, P. Levy, Mandelbrojt, Montel, Pérès, Platrier, Poncin, Roy, Thiry, Valiron, Vessiot, Villat.

To these names will be added at the time of publication in the Bulletin those friends and foreign colleagues of Maurice Fréchet who have assisted in obtaining subscriptions. Subscriptions should be sent to M. Daniel Dugué, Professeur a la Faculté des Sciences, 52, rue d'Authie, a Caen, Calvados.

Rules for the competition. Manuscripts in a foreign language should be accompanied by a typewritten summary in French. All manuscripts should be sent to the President of the Société Mathematique, Institut H. Poincaré, 11, rue Pierre Curie, Paris (5e) before March 1, 1950. They should treat of general analysis (theory of abstract spaces, transformations of abstract elements into abstract elements) or its applications.

Manuscripts should bear the name and address of the author. If preferred, authors who desire to remain anonymous can write an identifying sentence on the manuscript, which is reproduced on a sealed envelope containing the name and address of the author. Envelopes corresponding to memoirs which are not retained will be destroyed without being opened.

PERSONAL ITEMS

Dr. S. A. Schelkunoff of New York City has been awarded the Stuart Ballantine Medal of the Franklin Institute, Philadelphia, for his outstanding contributions to the extension of the electromagnetic wave theory.

Antioch College announces: Professor Max Astrachan, chairman of Department of Mathematics, who is on leave of absence during 1949–50, has been appointed Professor and Head of Department of Statistics, USAF Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio; Assistant Professor Parker Hamilton of Boston University has accepted an appointment at Antioch during the current academic year.

At Bowling Green State University: Dean Emeritus J. R. Overman has resumed full time teaching; Mr. H. E. Tinnappel of Ohio State University has been appointed to an assistant professorship; Mr. Irving Gaskill and Mr. Theodore Titgemeyer have resigned to take graduate work.

Case Institute of Technology announces the following: Associate Professor Max Morris has been promoted to a professorship; Mr. F. C. Leone, formerly instructor at Purdue University, has been appointed to an instructorship.

Centenary College makes the following announcements: Mrs. Fariebee P. Self has been promoted to an assistant professorship; Mr. Charles Murrah has been appointed to an instructorship; Associate Professor W. C. Griffith has been elected a vice chairman of the Louisiana-Mississippi Section of the Association.

Central Missouri State College announces the promotion of Assistant Professor L. W. Akers to an associate professorship and Instructor I. A. Gladfelter to an assistant professorship.

At Clark University: Professor C. E. Melville has retired with the title of Professor Emeritus; Mr. R. R. Christian, formerly graduate student at Yale University, has been appointed to an instructorship.

Connecticut College for Women reports: Mr. W. E. Ferguson, graduate as-

sistant at Yale University, has been appointed to an instructorship; Miss Elizabeth Hahnemann, teaching assistant, has returned to the University of Minnesota for further graduate study.

Fenn College announces the following promotions and appointments: Associate Professors K. D. Kelly and W. R. Van Voorhis have been promoted to professorships; Assistant Professor C. W. Topp has been promoted to an associate professorship; Dr. E. E. Haskins who has been located at Wright Field, Dayton, has been appointed to an associate professorship.

Hope College announces the promotions of Instructors J. E. Folkert and C. A. Steketee to assistant professorships.

At Iowa State College: Assistant Professor R. E. Gaskell has been promoted to an associate professorship; Instructors H. D. Block and H. E. Dickey have been promoted to assistant professorships; Dr. Carl Langenhop of Princeton University and Dr. J. E. Foote of Massachusetts Institute of Technology have been appointed to assistant professorships; Mr. W. S. Bicknell, graduate assistant at University of Michigan, and Mr. R. S. Dean, graduate student at North Texas State Teachers College, have been appointed to instructorships; Professor I. F. Neff and Assistant Professor Fay Farnum have retired.

Iowa State Teachers College announces the appointments of Mr. I. H. Brune of Frostburgh State Teachers College to an associate professorship and Mr. F. W. Lott, Jr., formerly student at University of Michigan, to an assistant professorship.

Macalester College reports: Mr. Robert Scherer, graduate assistant at University of Minnesota, has been appointed to an assistant professorship; Assistant Professor M. D. Brown is on leave of absence and is doing graduate work at the University of Minnesota.

North Central College announces the promotion of Assistant Professor Mary A. Seybold to a professorship and the appointment of Mr. H. G. Beck of the Avery Coonley School to an assistant professorship.

At North Texas State College: Assistant Professor J. V. Cooke has been promoted to an associate professorship; Mr. H. C. Parrish of Ohio State University has been appointed to an assistant professorship.

Pomona College makes the following announcements: Professor A. J. Kempner of the University of Colorado has been appointed to a professorship for the year 1949–50; Mr. C. J. A. Halberg has been appointed to an instructorship; Miss Jean B. Walton, who has been appointed Dean of Women, will teach part-time in the Department of Mathematics; Professor H. J. Hamilton is on leave for 1949–50 to teach at Los Angeles City College; Professor Elmer Tolsted is teaching at Exeter College, Devon, England during 1949–50.

Stanford University reports: Professor Kurt Reidemeister, University of Marburg, Germany, and Institute for Advanced Study, and Dr. P. D. Lax of New York University were visiting members of the staff during the Summer Quarter, 1949. Lectures were given by the following people during the summer: Professor Stefan Bergman, Harvard University; Professor K. Mahler, Univer-

sity of Manchester, England; Professor L. V. Ahlfors, Harvard University; Professor Arne Beurling, Sweden; and Professor Witold Hurewicz, Massachusetts Institute of Technology.

Nebraska State Teachers College, Wayne, announces that Mr. M. J. Hassel has been elected President of the Nebraska Section of the National Council of Teachers of Mathematics.

Trinity College announces the following: Mr. A. J. Grace, Jr., has been appointed to an instructorship; Mr. C. S. Ogilvy is on leave of absence and is doing graduate work at Columbia University.

Union College makes the following announcements: Associate Professors O. J. Farrell and A. H. Fox have been promoted to professorships; Mr. M. R. Bates, formerly teaching fellow at Cornell University, has been appointed to an instructorship; Mr. Edward Craig is now a graduate student at Massachusetts Institute of Technology.

At the University of Arizona: Dr. Deonisie Trifan of Case Institute of Technology and Dr. B. C. Meyer of Stanford University have been appointed to assistant professorships.

University of Washington reports the following: Dr. D. G. Chapman of the University of California has been appointed to an assistant professorship; Mr. D. M. Sandelius of the University of Uppsala has been appointed Lecturer; Assistant Professor Edward Paulson is on leave of absence to continue research at Columbia University.

Upsala College announces: Dr. Louis Larriver of the Naval Observatory has been appointed to an associate professorship; Mr. Donald Lindtredt, formerly instructor at the United States Naval Academy, has been appointed to an assistant professorship; Assistant Professor Norma M. Gilbert has resigned to continue graduate work.

Assistant Professor Leonard Bristow of the University of Wyoming has been appointed Head of Department of Mathematics of Wisconsin State Teachers College, Oshkosh.

Professor H. E. Buchanan, Head of the Department of Mathematics of Tulane University, has retired.

Assistant Professor Virginia Carlton of Centenary College has been appointed to an associate professorship at Northwestern Louisiana State College.

Dr. Uttam Chand has been appointed Assistant Professor of Mathematical Statistics at Boston University.

Assistant Professor George Cook of the Colorado School of Mines has been promoted to an associate professorship.

Mr. P. C. Cox, formerly graduate student at the University of Michigan, has been appointed to an assistant professorship at Albion College.

Mr. H. T. Donohoe of Baylor University has been appointed to an assistant professorship at Louisiana College.

Mr. J. D. Edwards has been appointed to an assistant professorship at Howard College.

Dr. Arthur Erdélyi of the University of Edinburgh has been appointed to a professorship at the California Institute of Technology.

Mr. Milford Franks, Dean of Men at Carthage College, is now teaching part-time in the Department of Mathematics.

Assistant Professor W. J. Fuchs of Cornell University has been promoted to an associate professorship.

Mr. G. R. Glabe, formerly instructor in the Galesburg Division of the University of Illinois, has been appointed to an instructorship at Denison University.

Mr. W. W. Gorsline of Wright Junior College has been appointed to an instructorship at Butler University.

Mr. R. T. Gregory, formerly at the United States Naval Proving Ground, Dahlgren, Virginia, has been appointed to an instructorship at Florida State University.

Miss Clara L. Hancock has retired from her position at Virginia Junior College, Minnesota.

Professor Mabel Heren of Knox College has retired with the title of Professor Emerita.

Mr. Edward Hodson of the University of Wisconsin, Milwaukee Branch, has been appointed Instructor of Mathematics and Physics at Cornell College.

Mr. R. T. Hood has been appointed to an instructorship at Beloit College.

Professor J. M. Kindle of the University of Cincinnati has retired with the title of Professor Emeritus.

Miss Elizabeth C. Kleinhans has been appointed to an instructorship at Illinois Institute of Technology.

Mr. D. V. LaFrenz of William Jewell College has been promoted to a professorship.

Mr. J. C. Lanz has been appointed a member of the Department of Mathematics of Hershey Junior College.

Assistant Professor Mary A. Lee of Sweet Briar College has been promoted to an associate professorship.

Professor R. R. McDaniel of Virginia State College has been appointed Acting Dean.

Professor W. R. McEwen of the University of Minnesota, Duluth Branch, has been promoted to an associate professorship.

Mr. V. D. Moore of Indiana State Teachers College, Terre Haute, has been promoted to an assistant professorship.

Assistant Professor Karlem Riess of Tulane University has been promoted to an associate professorship in the Department of Physics.

Mr. Arnold Ritchie has been appointed Supervisor of Mathematics in the Demonstration School of Central State College.

Associate Professor Helen G. Russell of Wellesley College has sabbatic leave for the first semester of 1949–50.

Instructor D. L. Shell of Michigan College of Mining and Technology has been granted a research fellowship at the University of Cincinnati.

Mr. J. L. Slechticky, University of Toledo, has received an appointment at New Mexico Highlands University.

Associate Professor H. E. Stelson is on leave of absence from Michigan State College; he has a position as visiting professor at the University of Hawaii.

Associate Professor Roscoe Stinetorf of Catawba College has been appointed Head of the Department of Physics of Wagner College.

Mr. A. D. Talkington of the University of Missouri has been appointed to an instructorship at DePauw University.

Dr. L. V. Toralballa of Fordham University has been appointed to an associate professorship at Marquette University.

Assistant Professor R. M. Whitmore of Southwestern University has been promoted to an associate professorship.

Professor C. E. Wilder of Dartmouth College has retired.

Assistant Professor Mary E. Williams of Skidmore College has been promoted to an associate professorship.

Miss Zung-nyi Loh, formerly lecturer at Wellesley College, has been appointed Assistant Professor of Physics at Wilson College.

Professor R. E. Gleason of Temple University died on July 7, 1949.

Assistant Professor G. F. Kelsall of Upsala College died on November 30, 1948.

Professor Nelle Miller of the University of Arizona died on June 20, 1949.

Dr. R. G. D. Richardson, dean emeritus of the Graduate School of Brown University, died July 17, 1949. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 80 persons have been elected to membership by the Board of Governors on applications duly certified:

M. R. BATES, A.B. (Cornell) Teaching Fellow, Cornell University, Ithaca, N. Y.

C. M. BELL, B.S. (Franklin & Marshall) Chemist, Naval Ammunition & Net Depot, Seal Beach, Calif.

C. E. BLOM, Editor, "Elementa," Stockholm, Sweden.

C. F. BRIGGS, M.A. (Michigan) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

BAILEY BROWN, M.A. (Princeton) Professor, Amherst College, Mass.

ELIZABETH W. BROWNELL, B.A. (Vassar) Student, Stanford University, Calif.

G. C. BURNS, B.S. (Oklahoma A & M) Grad. Fellow, Oklahoma A & M, Stillwater, Okla.

L. J. BURTON, Ph.D. (Harvard) Asst. Professor, Bryn Mawr College, Pa.

E. F. BUSCH, Student, St. Norbert College, West De Pere, Wis.

L. H. CHAMBERS, Ph.D. (Cornell) Assoc. Professor, U. S. Naval Academy, Annapolis, Md.

W. J. CHERRY, M.A. (Northwestern) Teacher, Morton High School and Junior College, Cicero, Ill.

K. G. CLEMANS, M.A. (Minnesota) Instructor, Willamette Univerity, Salem, Ore.

REGINAL COBB, Student, University of Florida, Gainesville, Fla.

W. W. COLEMAN, B.A. (Cornell) Asst. Vice President, Irving Trust Co., New York, N. Y.

NICOLAS COLMENARES-CARRILLO, Ing. Civ. (Univ. Nac. de Colombia) Engineer, Caracas, Venezuela.

L. E. DAVIS, JR., B.Sc. (Ohio State) Research Asst., Ohio State University, Columbus, Ohio.

MRS. B. PEARSEN DELANY, B.S. (Illinois Institute of Technology) Student, Illinois Institute of Technology, Chicago, Ill.

H. T. DONOHOE, M.A. (Baylor) Asst. Professor, Louisiana College, Pineville, La.

H. L. EMERSON, Jr., B.S. (Beloit) Student, Beloit College, Wis.

PAUL ERDÖS, D.Sc. (Manchester) Visiting Research Professor, Syracuse University, N. Y.

A. R. ERSKINE, M.A. (Michigan) Instructor, Pennsylvania State College, DuBois, Pa.

E. R. FINKBEINER, Student, St. Norbert College, West De Pere, Wis.

MARY J. FLANARY, B.A. (St. Teresa) Teacher, Academy of Holy Angels, Minneapolis, Minn.

CHARLES FRANCIOL, B.S. (Southwestern Louisiana Institute) Grad. Asst., Louisiana State University, Baton Rouge, La.

JACQUELINE GIVEN, M.A. (Colorado) Head of Department, Pueblo Junior College, Colo.

G. R. GLABE, M.A. (Minnesota) Instructor, Denison University, Granville, Ohio.

R. R. GUTZMAN, M.S. (Iowa) Instructor, Fenn College, Cleveland, Ohio.

D. L. GUY, B.A. (Macalester) Wauwatosa, Wis.

C. J. A. HALBERG, JR., B.A. (Pomona) Instructor, Pomona College, Claremont, Calif.

MR. DILLA HALL, M.S. (Chicago) Asst. Professor, Southern Illinois University, Carbondale, Ill.

HELEN J. HAND, M.S. (Fordham) Instructor, D'Youville College, Buffalo, N. Y.

P. C. HANZEL, B.S. (Duquesne) Grad. Asst., University of California, Berkeley, Calif.

W. R. HARRIS, JR., LL.B. (Southern Methodist) Attorney, Dallas, Texas.

A. S. HENDLER, M.A. (Columbia) Instructor, Rensselaer Polytechnic Institute, Troy, N. Y.

A. F. HERBST, M.A. (Maryland) Asst. Professor, La Verne College, Calif.

H. G. HERTZ, Ph.D. (Yale) Assoc. Astronomer, U. S. Naval Observatory, Washington, D. C.

D. M. HESTER, B.A. (Southern Methodist) Asst. Professor, Baker University, Baldwin City, Kansas.

ROBERT HOOKE, Ph.D. (Princeton) Assoc. Professor, University of the South, Sewanee, Tenn.

I. M. HOSTETTER, Ph.D. (Washington) Assoc. Professor, Oregon State College, Corvallis, Ore.

L. AILEEN HOSTINSKY, Ph.D. (Illinois) Instructor, Temple University, Philadelphia, Pa.

D. W. HULLINGHORST, B.S. (Tulane) New Orleans, La.

C. J. KAUFMAN, Student, New York, N. Y.

G. W. KAYS, M.A. (Montclair) Instructor, Montclair Teachers College, N. J.

W. D. KRENTEL, B.A. (Louisiana Polytechnic Institute) Grad. Fellow, Oklahoma A & M, Stillwater, Okla.

CECILIA KRIEGER, Ph.D. (Toronto) Asst. Professor, University of Toronto, Ont.

W. M. LAIDLAW, Student, Willamette University, Salem, Ore.

R. E. LEE, M.S. (Missouri) Asst. Professor, Missouri School of Mines and Metallurgy, Rolla, Mo.

W. T. LENSER, Sc.M. (Brown) Instructor, University of Nebraska, Lincoln, Neb.

TADEUSZ LESER, Ph.D. (London) Asst. Professor, University of Kentucky, Lexington, Ky.

ZUNG-NYI LOH, M.A. (Cornell) Asst. Professor, Wilson College, Chambersburg, Pa.

G. H. MACCULLOUGH, Sc.D. (Michigan) Professor, Worcester Polytechnic Institute, Mass.

F. H. MCGAR, JR., B.A. (Yale) Instructor, Fenn College, Cleveland, Ohio.

C. R. MCINTOSH, B.S. (Holy Cross) Instructor, St. Thomas College, St. Paul, Minn.

J. M. MCLYNN, Student, George Washington University, Washington, D. C.

W. W. MITCHELL, JR., M.A. (Colorado) Instructor, Phoenix College, Ariz.

J. C. MORELOCK, M.A. (Missouri) Teaching Asst., University of Florida, Gainesville, Fla.

C. H. MURPHY, JR., M.A. (Johns Hopkins) Junior Instructor, Johns Hopkins University, Baltimore, Md.

R. F. NEWELL, B.S. (South Carolina) Junior Instructor, New York State Institute of Applied Arts and Sciences, Buffalo, N. Y.

J. A. NICKEL, B.S. (Willamette) Student, Oregon State College, Corvallis, Ore.

J. S. NODVIK, Student, Carnegie Institute of Technology, Pittsburgh, Pa.

EDNA M. NORSKOG, M.A. (Columbia) Instructor, Illinois State Normal University, Ill.

J. M. PATTERSON, A.M. (Columbia) Instructor, Wayne University, Detroit, Mich.

D. J. PETERSON, B.A. (Occidental) Student, Occidental College, Los Angeles, Calif.

L. D. POTTS, B.M.E. (Ohio State) Research Engineer, Linde Air Products Co., Tonawanda, N. Y.

H. W. REIMER, Student, Long Island University, Brooklyn, N. Y.

D. D. RIPPE, M.A. (Nebraska) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

ALEX ROSENBERG, M.A. (Toronto) Fellow, University of Chicago, Ill.

L. L. ROSS, M.A. (Ohio State) Instructor, Ohio Northern University, Ada, Ohio.

S. M. SHARTLE, Surveyor, Office of County Surveyor, Danville, Ind.

ELYSE G. SHEPPARD, M.A. (Michigan) Asst. Professor, University of Tampa, Fla.

G. W. STARCH, Student, Oklahoma A & M, Stillwater, Okla.

MR. DEONISIE TRIFAN, Ph.D. (Brown) Asst. Professor, University of Arizona, Tucson, Ariz.

SUSIE L. WARD, M.A. (Alabama) Instructor, University of Alabama, University, Ala.

MARY E. WILCOX, M.A. (Southwestern) Asst. Professor, Southwestern University, Georgetown, Texas.

ANNIE J. WILLIAMS, M.A. (North Carolina) Teacher, Julian S. Carr Junior High School, Durham, N. C.

W. H. WITTY, B.A. (Mississippi) Winona, Miss.

ALBERT WOLINSKY, Ph.D. (Vienna) Instructor, New York University, N. Y.

VERBA M. WOOD, B.S. (Roanoke) Instructor, College of William & Mary, Williamsburg, Va.

CHIA-SHUN YIH, Ph.D. (Iowa) Lecturer, University of British Columbia, Vancouver, B. C.

LILLIAN B. ZARLING, M.A. (Minnesota) Instructor, University of Wisconsin, Green Bay, Wis.

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-ninth regular meeting of the Southern California Section of the Mathematical Association of America was held at John Muir College in Pasadena, California on Saturday, March 12, 1949. Professor E. F. Beckenbach, Chairman of the Section, presided at the morning session and at one of the afternoon sessions; Professor H. R. Pyle, Vice-Chairman of the Section, presided at the other afternoon session.

The attendance was one hundred thirty-five, including the following sixty-five members of the Association: L. J. Adams, H. M. Bacon, E. F. Beckenbach, May M. Beenken, H. F. Bohnenblust, Herbert Busemann, F. A. Butter, Jr.,

W. D. Cairns, B. A. Chiappinelli, P. A. Clement, L. M. Coffin, Myrtie Collier, E. L. Crow, D. R. Curtiss, J. H. Curtiss, P. H. Daus, R. P. Dilworth, W. H. Glenn, Jr., B. K. Gold, Jr., H. J. Hamilton, R. B. Herrera, M. R. Hestenes, E. J. Hills, P. G. Hoel, R. E. Horton, D. H. Hyers, C. G. Jaeger, Glenn James, G. R. Kaelin, Samuel Karlin, P. J. Kelly, Cornelius Lanczos, Ella E. Lausman, L. C. Lay, Margaret B. Lehman, Sophia L. McDonald, G. F. McEwen, W. E. Milne, F. R. Morris, W. H. Myers, J. F. Paris, R. P. Peterson, Jr., H. R. Pyle, E. D. Rainville, L. T. Ratner, L. L. Rauch, E. C. Rex, Wladmir Seidel, G. E. F. Sherwood, Ernst Snapper, R. H. Sorgenfrey, I. S. Sokolnikoff, D. V. Steed, J. D. Swift, T. E. Syndor, A. E. Taylor, W. I. Thompson, C. W. Trigg, S. E. Urner, F. A. Valentine, Morgan Ward, R. L. White, A. L. Whiteman, D. V. Widder, Euphemia R. Worthington.

At the business meeting the following officers were elected for the next academic year: Chairman: H. R. Pyle, Whittier College; Vice-Chairman, Herbert Busemann, University of Southern California; Program Committee, H. F. Bohnenblust (Chairman), Lulu Bechtolsheim, and R. E. Horton; and Secretary, P. H. Daus, ex-officio.

The next meeting was scheduled for March 11, 1950 at Immaculate Heart College, Hollywood, California.

The following papers were presented:

1. *The theory of games*, by Professor H. F. Bohnenblust, California Institute of Technology.

The fundamental notions of the theory of two-person, zero-sum games were reviewed, and the main points illustrated by a simple example. A discussion of the material was led by Mr. L. S. Shapley and Dr. S. Karlin. Mr. Shapley discussed the extension to infinite games, stressing in particular the difficulties which are encountered when the pay-off function is not suitably restricted. Dr. Karlin spoke on the case of continuous games in which the pay-off function is a polynomial. The relationship to the moment problem was discussed, and attention was called to some of the more important unsolved problems.

2. *Divergence in mathematical definitions and concepts*, by Professor C. G. Jaeger, Pomona College.

This speaker pointed out that there is considerable lack of agreement in definition, usage, and interpretation of many mathematical concepts. It was suggested that some national or international mathematical body attempt to bring about a standardization in cases where it is needed; and that the result be used as an authority where differences of opinion occur.

3. *On L-sets*, by Professor F. A. Valentine, University of California at Los Angeles.

A set S is called an L_n set if each pair of points in S can be joined by a polygonal line having at most n segments. Properties of L_2 sets developed by Professor A. Horn and the author were described. The relationship between L_n sets and C_n-convex sets was discussed. A C_n-convex set is one having n components, each of which is convex. The complement of an open, bounded C_n-convex set ($n>1$) is an L_{n+1} set. To prove this the author uses the notion of a maximal family of n disjoint open convex sets.

4. *Mortality statistics*, by Dr. J. L. Brenner, Santa Barbara College, University of California, introduced by Professor Herbert Busemann.

This paper gives a general description of mortality statistics, and an analysis of the monetary functions based on them. Some of the principal mortality tables are examined; "selection" is mentioned, and the underlying causes of the existence of a "select period" are brought out.

The provisions, and especially the cost, of a retirement plan, will usually change with the mortality of a group of lives covered by the plan. Several published tables purport to predict the mortality of a group of retired lives. The predictions of these tables are mutually inconsistent. This paper mentions a method which is being used for gathering, and analyzing the reliability, of mortality statistics for retired university teachers. The novelty of the formula used to compute the reliability of the statistics lies in the point of view involved in its application.

5. *Local motions and non-euclidean geometry*, by Dr. P. J. Kelly, University of Southern California.

Given a finite portion of a two dimensional metric space, in which geodesics exist uniquely, the extra assumption that neighborhoods possess reflections onto themselves specializes the geometry to being hyperbolic, elliptic, or euclidean.

6. *A miniature theory in illustration of the convolution transform*, by Professor D. V. Widder, Harvard University.

The convolution transform is defined by the equation

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt.$$

For a large class of "kernels" $G(x)$, the author and I. I. Hirschman, Jr., have discussed elsewhere the inversion of this transform, that is, the determination of $\phi(t)$ from $f(x)$. They have shown in fact that the inversion is accomplished by means of a linear differential operator, of infinite order, with constant coefficients. In the present paper this general theory is illustrated by use of the special kernel $G(x)=e^x$, $(x<0)$; $G(x)=0$, $(x>0)$. In this case many of the results take the same form as in the general case, but can be established *ab initio*. Since the Laplace and Stieltjes transforms are special cases of the convolution transform, they are also illustrated by the present miniature theory.

7. *Remainder term in linear methods of approximation*, by Professor W. E. Milne, Oregon State College and the Institute for Numerical Analysis.

This paper represents an effort to formulate a systematic treatment of the error for such diverse processes as interpolation, numerical differentiation, numerical integration, harmonic analysis, approximation by least squares, approximation by equating moments, and other allied operations.

First a uniform procedure is exhibited by which a desired method of approximation may be explicitly constructed. Second, by suitable transformation the remainder is put in a form in which it is possible in many cases to estimate its approximate magnitude. And third, the theory is applied to a variety of concrete examples, and bounds are obtained for the magnitude of the error.

8. *Possibility and necessity*, by Dr. J. C. C. McKinsey, Douglas Aircraft Corporation, introduced by Professor E. F. Beckenbach.

The speaker outlined the results in modal logic obtained during the last three decades by C. I. Lewis, W. T. Perry, M. Wajsberg, K. Godel, J. C. C. McKinsey, R. Carnap, and others. Considerable attention was devoted to the formal properties of the five systems of modal logic described by C. I. Lewis.

9. *Report of the state-wide Mathematical Education Committee*, by Professor Frank Morris, Fresno State College, and Professor H. M. Bacon, Stanford University.

P. H. DAUS, *Secretary*

THE APRIL MEETING OF THE MICHIGAN SECTION

The spring meeting of the Michigan Section of the Mathematical Association of America was held in conjunction with the meeting of the Michigan Academy of Science, Arts, and Letters at Wayne University, Detroit, Michigan, April 2, 1949. This meeting also constituted the meeting of the Mathematics Section of the Michigan Academy of Sciences, Arts, and Letters. Morning and afternoon sessions and a luncheon-business meeting were held, at all of which Professor B. M. Stewart, Chairman of the Section, presided.

About seventy-five people attended the meeting including the following fifty-two members of the Association: Bess E. Allen, N. H. Anning, J. W. Baldwin, W. D. Baten, F. A. Beeler, J. H. Bell, C. J. Blackall, W. M. Borgman, J. W. Bradshaw, J. R. Britton, D. M. Brown, R. E. Carr, R. V. Churchill, B. B. Clark, W. H. Clatworthy, P. C. Cox, J. W. Crispin, P. S. Dwyer, P. W. Edmonson, K. W. Folley, J. S. Frame, E. L. Grindall, G. W. Grotts, V. G. Grove, Fritz Herzog, T. H. Hildebrandt, J. D. Hill, E. E. Ingalls, L. G. Johnson, L. S. Johnston, Wilfred Kaplan, H. D. Larsen, L. E. Mehlenbacher, E. D. McCarthy, J. A. McGrail, D. C. Morrow, A. L. Nelson, Mary H. Payne, Gertrude V. Pratt, G. Y. Rainich, P. H. Raker, L. E. Shaefer, Tryphena H. Scibiorski, W. F. Smith, T. H. Southard, R. L. Spencer, H. E. Stelson, B. M. Stewart, P. C. Sweetland, L. O. Thompson, W. R. Utz, J. E. Vollmer.

The following officers were elected for the coming year: Chairman, Professor L. E. Mehlenbacher, University of Detroit; Secretary-Treasurer, Professor P. S. Jones, University of Michigan.

At the morning and afternoon sessions the following program of papers was presented:

1. *Statistics as part of a general education in mathematics*, by Benjamin Epstein, Wayne University.

It is becoming apparent that an understanding of some of the basic concepts in probability and statistics is essential to the solution of many scientific problems. This means, in the opinion of the speaker, that some basic changes may have to be made in the undergraduate mathematics curriculum. These changes may well require a breaking away from traditional concepts of what constitutes elementary mathematics.

2. *A continued fraction formula for the remainder in a certain series*, by Professor Emeritus J. W. Bradshaw, University of Michigan.

The author finds a formula for the remainder in the special case of hypergeometric series $F(p, p; 1; 1)$ after summing a fixed number of terms. It is expressed as the product of the last term taken and a certain continued fraction. It not only provides a means of rapid calculation, comparable to that of the sum in terms of gamma functions $\Gamma(1-2p)/\Gamma^2(1-p)$, but also yields an extension of the series outside its interval of convergence.

3. *Note on the product of power sums*, by Professor J. S. Frame, Michigan State College.

If $S_p(n)$ denotes the sum of the pth powers of the integers from 1 to n, then it is well known that $S_p(n)$ is a polynomial in n of degree $p+1$, whose coefficients are simply expressible in terms of Bernoulli numbers. The relation $S_1^2(n) = S_3(n)$ is also familiar. In this paper relations such as $2S_3^2(n) = S_7(n) + S_3(n)$ are included in a general formula expressing the product $S_p(n)\,S_q(n)$ of two power sums as a linear combination of the power sum polynomials $S_{p+q+1}(n)$, $S_{p+q-1}(n)$, $\cdots$, in which the sum of the coefficients is unity, the subscripts $p+q+1$, $p+q-1$, etc. all have the same parity, and the number of terms is $1+[p/2]$, if $p \geqq q$. The proof is obtained by summing both sides of the identity

$$S_p(m)S_q(m) - S_p(m-1)S_q(m-1) \equiv m^p[S_q(m) - m^{q/2}] + m^q[S_p(m) - m^{p/2}]$$

from $m=1$ to n.

4. *The number systems of algebra and some trouble spots in grade school arithmetic*, by Professor Holmes Boynton, Northern Michigan College of Education, introduced by the Secretary.

The five number systems of algebra (natural, rational or fractional, signed, real, complex) have been shown to be built one on the other in order to satisfy a logical need; the rational number system to create a system closed to division; the signed, closed to subtraction; the real, closed to the process of taking roots of positive numbers; and the complex, closed to the process of taking roots of negative numbers; and to form a system capable of expressing roots of any algebraic equation.

The logical steps involved in creating each system from the previous one, and of showing the previous one isomorphic to a subset of the new, are not easy for a college student. The grade school child, when first confronted with manipulation of fractions, signed numbers, and real numbers (in the form of square roots, and of π) is often similarly confused. The teacher too often progresses from one system to another without knowing what she is doing, or that difficulties are involved which may confuse the gifted child as well as the normal one.

It is, however, possible for a grade school teacher to develop fractions and signed numbers in a manner understandable to her pupils by analyzing the types of situations in which the need for such numbers arises. This will lead the child to associate in his mind the kinds of number with various things he does. But these things are analogous to the reasons which induce the mathematician to develop the systems in a logical manner.

5. *Some remarks concerning pattern integration*, by Professor R. E. Carr, Michigan State College.

When investigating the asymptotic behavior of a number theoretic function such as

$$H(n) = \sum_{i,j}^{n} i^3 j^3, \qquad (i + 2j = n)$$

the substitution $x_i = 1/n$ leads to

$$n^{-6}H(n) = \sum_{\substack{i=1 \\ (p)}}^{n-2} x_i^3 \left(\frac{1 - x_i}{2}\right)^3 (x_i - x_{i-1})$$

where $P: i \equiv n \bmod 2$ indicates that the summation is to take place over a subset of the set $\{i\}_1^{n-2}$. The result,

$$\lim_{n\to\infty} [n^{-6}H(n)] = \frac{1}{2}\int_0^1 x^3 \left(\frac{1-x}{2}\right)^3 dx$$

does not follow directly from the definition of the Riemann integral. If to the Riemann integral,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),$$

is added the restriction that the summation is to take place over a certain subset P of the set $\{i\}^\infty$, the resulting limit, providing it exists, will be called the P-pattern integral of $f(x)$. This paper contains some remarks and a few theorems about pattern integration.

6. *A multiplicative Diophantine equation*, by Gerald Harrison, Wayne University, introduced by the Secretary.

Some consequences, especially in the ring of Gaussian integers, of the general solution of the Diophantine equation $xy=zw$ were discussed.

7. *Theory of tests of statistical hypotheses*, by C. W. Churchman, Wayne University, introduced by the Secretary.

After brief references to the work of Pearson, Neyman, and Wald on the theory of criteria for "best" tests of statistical hypotheses, this speaker considers certain extensions of their work, and outlines the mathematical problems involved if these extensions are to be carried out to completion.

8. *The rate of interest in installment payment plans*, by Professor H. E. Stelson, Michigan State College.

Three formulas (besides the compound interest formula) are commonly used in determining the rate of interest in installment payment plans. These are known as the constant ratio, series of payments, and interest at end formulas. These formulas have been independently developed from certain particular assumptions. It is the purpose of this paper to show that all three of these formulas may be derived as approximations to the compound interest formula (sometimes called the actuarial formula). It is also shown that the rates as determined by the various formulas satisfy definite inequalities. A new and more accurate approximation formula is presented.

9. *Mathematical statistics from the standpoint of elementary algebra*, by Professor H. W. Alexander, Adrian College, introduced by the Secretary.

Most of the available treatments of mathematical statistics presuppose an advanced knowledge of the calculus. This paper will show that the central ideas of mathematical statistics may be developed and illustrated using only algebraic ideas such as are encountered in the typical college algebra course. The central problems which are dealt with are these: (1) Given an infinite population with a discrete distribution, to derive the distributions of the means and variances of small samples; (2) Given a set of observed values, to set up and test a null hypothesis concerning the parent population from which they are assumed to be drawn.

10. *Clarification of the problem of damped free vibrations*, by Professor G. P. Loweke, Wayne University.

The solution of the problem of damped free vibrations is presented incorrectly in one respect or another in a surprising number of texts. No general solution of the problem is presented in any standard text, and speculation arises as to the exact nature of the motion. This paper is intended to clarify the misunderstanding which can arise from these solutions.

11. *The Green's function for the rectangle obtained by the finite Fourier transformation*, by A. W. Jacobson, Wayne University, introduced by the Secretary.

A special function which arises in the solution of boundary value problems is introduced, and

in terms of this function a new formula for Green's function for the rectangle is given. A relation is established between the special function and Weierstrass's sigma function.

L. J. ROUSE, *Secretary*

THE APRIL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The April meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the University of Mississippi, Oxford, Mississippi, April 8–9, 1949. Professor W. L. Duren, Chairman of the Section, presided.

Ninety-eight persons attended the meeting, including the following forty-six members of the Association: T. A. Bickerstaff, J. A. Bullard, L. Virginia Carlton, Margaret R. Davis, Mamie M. Davis, W. L. Duren, Virginia I. Felder, L. M. Garrison, M. E. Gillis, F. Monica Goen, W. C. Griffith, A. C. Grimes, R. H. Hopkins, Alfred Hume, W. R. Hutcherson, H. T. Karnes, C. G. Killen, Z. L. Loflin, Saunders MacLane, A. C. Maddox, J. W. McClimans, Dorothy McCoy, Betty McKnight, R. A. Miller, B. E. Mitchell, S. B. Murray, I. C. Nichols, Arthur Ollivier, C. R. Pettis, O. L. Phillips, P. K. Rees, F. A. Rickey, Alta H. Samuels, H. F. Schroeder, H. L. Smith, M. M. Temple, V. B. Temple, W. B. Temple, J. E. Thomson, B. B. Townsend, G. R. Trott, B. A. Tucker, P. M. Tullier, Eleanor B. Walters, K. L. Warren, M. C. Wicht.

The following officers were elected for the coming year: Chairman, G. R. Trott, University of Mississippi; Louisiana Vice-Chairman, W. C. Griffith, Centenary College; Mississippi Vice-Chairman, M. E. Gillis, Blue Mountain College; Secretary-Treasurer, F. A. Rickey, Louisiana State University.

By the invitation of the Executive Committee, Professor Saunders MacLane delivered two addresses, one at the Friday evening informal dinner on the subject *Mathematics in Europe*, the other at the Saturday morning session on *Determinants and Grassman algebras*.

The following short papers were presented:

1. *Third order involution contained on a certain seventh degree surface*, by W. R. Hutcherson, Northwestern State College of Louisiana.

This paper concerns a seventh degree surface which is invariant under the homography

$$T;\ x_1 : x_2' : x_3' = \epsilon x_1 : \epsilon^3 x_2 : x_3 : x_4, \qquad (\epsilon^3 = 1).$$

The equation of the surface is

$$F(x_1, x_2, x_3, x_4) \equiv a x_2^6 x_1 + b x_1 + c x_1^5 x_2 + d x_2^3 + e x_1^3 x_2 + f x_1 x_2^3 = 0,$$

where a, b, c, d, e, and f are homogeneous functions of x_3 and x_4 of degrees 0, 6, 1, 5, 4, and 3, respectively. This third order involution is mapped on a surface in a space of seven dimensions by use of suitable equations. This surface is of order 21, and the five equations defining the surface are

$$X_5^3 = X_1 X_2 X_3, \qquad \left\| \begin{matrix} X_3 X_2 X_7 X_5 \\ X_4 X_7 X_3 X_6 \end{matrix} \right\| = 0, \qquad \psi(X_1, X_2, X_3, X_4) = 0.$$

The seventh degree surface in three-way space has two isolated invariant points as well as a

line of invariant points. One of these coincident (invariant) points $P_3(0, 1, 0, 0)$, is found to be a perfect point. (A perfect point is one such that a curve through this point, and also its transformed curve, both pass through it in the same direction.)

2. *Undergraduate mathematics in our Louisiana-Mississippi Section*, by Dorothy McCoy, Belhaven College.

This paper is a report on results obtained from a questionnaire sent to the senior colleges of Louisiana and Mississippi. Twenty-one questions covering mathematics curricula, course content, and related topics were included.

3. *Note concerning transforms of Fuchsian groups*, by P. K. Rees, Louisiana State University.

This paper considers a Fuchsian group of transformations $T=(az+c)/(cz+d)$, a fixed Fuchsian transformation $G=(\alpha z+\beta)/(\gamma z+\bar{\alpha})$, and the transform $S=GTG^{-1}$ of the group. It is proved that if r_s and r_t are the radii of the isometric circles of S and T, and if g_t, g_t', and g are the centers of the isometric circles of T, T^{-1}, and G, and if $r_s\sqrt{k}=r_t$, then there are no relative maximum or minimum values of k, the absolute minimum is zero and is taken on if and only if the mid-points of the segments (g_t, g_t') and $(g, 1/\bar{g})$ coincide; furthermore, this value can be taken if and only if T is an elliptic transformation.

4. *Compound cycloids*, by B. E. Mitchell and K. L. Warren, Millsaps College.

A compound cycloid is defined as a curve resulting from the compounding of two circular motions; i.e., the rotation of a circle called the guide circle, of radius a, about its fixed center and that of the generating circle, of radius b, $b<a$, which rolls on the guide. The locus of any point on the generating circle will be a compound cycloid. Professors Mitchell and Warren selected four loci for investigation, two without and two within the guide, the two points in each case being diametrically opposite. The parametric equations and the properties of these loci were developed. One interesting result is that the lengths of the compound cycloids are all expressible as elliptic integrals. As the ratio a/b increases, the lengths of these curves approach that of the guide circle (which the lengths of the ordinary epi- and hypo-curves do not).

5. *Harmonic measure in one-to-one directly conformal mappings of canonical plane regions, preliminary report*, by R. W. Schmied, Tulane University, introduced by the Secretary.

This paper is a study of a special case. A sequence $\xi_n=f_n(z)$ of functions each of which is analytic in a region $R_n\subset N_1(0)$ and bounded by $C_1(0)$ and three circles, and which maps R_n in a 1–1 directly conformal manner onto a region $\Omega_n\subset N_1(0)$ is described, and the mapping effected by $\xi_0=f_0(z)=\lim_{n\to\infty} f_n(z)$ for each $z\in R=\lim R_n$ of R onto $\Omega=\lim_{n\to\infty}\Omega_n$ is discussed. Harmonic measure plays the dominant role in the analysis of the mappings considered.

6. *On the extensive derivative of absolute extensors*, by B. B. Townsend, Louisiana State University.

The process of extensive differentiation of absolute extensors introduced by Craig in *Mathematics Magazine*, vol. 21 (1947), pp. 21–29, also *Bulletin of American Mathematical Society*, vol. 53 (1947), pp. 332–343, is developed for other types of tensor-extensors. The extensive derivative based on asymmetric connections is considered, and the corresponding extended components of connection are found.

7. (a) *A universal distributive law*, by G. R. Trott, University of Mississippi.

This speaker considers the expression $\alpha\circ(x\square y)$, where $\circ$ and $\square$ are certain specified operations,

and α, x, and y (x and y in general contained in the same set) are elements of the same set or different sets. The condition

$$\alpha \circ (x \square y) = \alpha \circ x \square \alpha \circ y \tag{1}$$

is easily recognized as a type of distributive law. By assigning specific operations to $\circ$ and $\square$, such as multiplication, addition, intersection, union, linear transformation, etc., basic requirements are found for homomorphisms, representations, algebras, tensors, linear vector spaces, etc.

(b) Brief reports by three students of the University of Mississippi, introduced by Professor Trott: *A note on DeGua's Theorem*, by Frank Fant, giving a correction to a translation of a theorem of DeGua concerned with the exact number of complex roots of an equation with missing terms; *Some remarks on quasi groups*, by Evelyn Wright, an attempt to simplify the definitions of certain groupoides and domains in which the associative law of multiplication does not hold, using a continuance of the definitions of rings, integral domains, and fields as given by Dubriel wherein he characterizes the definitions by the use of demi-groups and semi-groups; *On moments and semi-invariants*, by S. R. Knox, a paper on some properties of generating functions of moments and semi-invariants.

8. *A note on the natural number system*, by H. L. Smith, Louisiana State University.

Let I be the class of all natural numbers, and, for each natural number n let $s(n)$ denote the "successor of n" as in Peano's foundation. Then without first defining addition, the relation $<$ can be introduced into I as follows. For each n in I let M_n denote the class of all subsets E of I satisfying the following conditions: (1) $1 \in E$, (2) $s(E) \leqq E + [n]$. (Here $s(E)$ denotes the class of all $s(n)$ for n in E). Let I_n be the intersection of the sets of M_n. Then the relation $M < n$ may be defined to mean $m \neq n$, $I_m \subset I_n$.

9. *Report on the testing program*, by H. T. Karnes, Louisiana State University.

The departments of mathematics in the colleges and universities of Louisiana and Mississippi have instituted a testing program for entering Freshmen. This number on the program was a report of the results of the testing for the 1948–49 school year. It was felt that the results were of sufficient use and importance to justify continuation of the program for another year at least.

F. A. RICKEY, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held in Denton, Texas, April 8–9, 1949. North Texas State Teachers College and Texas State College for Women were co-sponsors.

The attendance was ninety-one, including the following forty-one members of the Association: A. W. Ashburn, A. E. Barksdale, Ina M. Bramblett, H. E. Bray, Myrtle C. Brown, J. E. Burnam, L. A. Colquitt, J. V. Cooke, Don Cude, J. A. Daum, R. E. Greenwood, E. H. Hanson, H. M. Hardy, C. M. Howard, J. M. Hurt, Mrs. Fay Johnson, E. C. Klipple, C. G. Maple, Hazel L. Mason, E. K. McLachlan, W. K. McNabb, Harlan C. Miller, Edith L. Morgan, E. D. Mouson, Jr., C. A. Murray, J. W. Querry, Henry Rainbow, L. W. Ramsey, C. R. Sherer, D. P. Shore, M. M. Slotnick, F. W. Sparks, D. W. Starr, Jennie L. Tate, H. E. Taylor, F. E. Ulrich, Maud Willey, Mabel Williams, H. A. Wood, L. G. Worthington, C. B. Wright.

At the business meeting the following officers were elected for the next academic year: Chairman H. J. Ettlinger, University of Texas; Vice-Chairman C. A Murray, West Texas State Teachers College; Secretary-Treasurer C. R. Scherer, Texas Christian University.

A banquet was given Friday night in the Household Arts Building on the campus of the Texas State College for Women. Dr. Floyd Stovall of North Texas State Teachers College was the principal speaker. A luncheon was given Saturday noon in Marquis Hall on the campus of North Texas State Teachers College.

The first three papers were read Friday afternoon, the next four papers Saturday morning, and the panel discussion was held Saturday afternoon.

1. *Regions of flatness of analytic functions*, by F. E. Ulrich, The Rice Institute.

At the St. Louis meeting of the American Mathematical Society in November 1947, there were given some results of a joint paper by Mandelbrojt and Ulrich concerning regions of flatness of analytic functions. At that time a definition of regions of flatness was stated, and conditions given under which the regions of a set will be regions of flatness for the derivative when they are regions of flatness for the function. For a statement of these results see Abstract No. 58 in the *Bulletin of the American Mathematical Society*, vol. 54 (1948).

In the present paper conditions are stated under which the regions of a set will be regions of flatness for all the derivatives when they are regions of flatness for the function. For a detailed statement of these results see Szolem Mandelbrojt and Floyd Ulrich, *Sur les domaines de comportement uniforme d'une fonction analytique*, Compte Rendus Acad. Sci. Paris, t. 226 (1948), pp. 152–153.

2. *Paths of minimum flight time*, by L. A. Colquitt, Texas Christian University.

The speaker dealt with the problem of minimizing the time of flight, if airspeed is constant and the wind distribution is given. In previous treatments, beginning with that by Zermolo for the plane, the methods were those of the calculus of variations. It was shown that if the formulation is changed (in a way consonant with navigational practice) the problem becomes one in minimizing a function of several variables.

3. *Exponential numerical integration*, by R. E. Greenwood, University of Texas.

A method of numerical integration for functions which can be expanded in powers of e^x has been investigated. Numerical tabulation of the coefficients in the integration formulae include sixth powers in one case, and ranges from negative third to positive third powers of e^x in a second case. Error expressions have also been found. The method has possibilities for use in exponential growth and exponential decay situations.

4. *Concerning certain topics in the calculus*, by H. E. Bray, The Rice Institute.

The speaker discussed the question "How should calculus be taught." He took the view that, on account of its usefulness in science and engineering, calculus should be begun early, that is, in the freshman year; so that by the end of that year the student may be fairly familiar, at least intuitively, with the basic ideas. From that point on calculus should be taught systematically, with emphasis on its logical structure, in order that the student may carry away a thorough under-

standing of the main body of theorems which constitutes the essential structure of the subject. The speaker gave point to his ideas by means of an outline of the central theorems taken from such topics as the theory of limits, continuous functions, derivatives and differentials, anti-derivatives (primitives), definite integrals, and various types of mean value theorems and their applications. It was contended that the more systematic course, of the kind proposed, if skillfullly conducted with an adequate textbook, can be made beneficial and interesting, not only to talented students but to those of average ability.

5. *Polynomials*, by G. R. MacLane, The Rice Institute.

Mr. MacLane discussed the usefulness of the representation

$$f(z) = \frac{1}{2\pi i}\int \frac{g(t)dt}{t-z}$$

where $g(t)$ is not necessarily $f(t)$. His ideas were illustrated by the connection with the following theorem: If $f(z)$ is analytic and non-vanishing in D, the interior of a rectifiable Jordan curve, then there exists a sequence of polynomials $P_n(z)$, all of whose zeros on Γ, such that line $P_n(z) = f(z)$, the convergence being uniform in any closed subset of D.

6. *A Mongean projection board*, by E. K. McLachlan, Baylor University.

The speaker exhibited a Mongean projection board consisting of three mutually perpendicular planes showing each of the eight octants. This projection board, made from plexiglass, was designed primarily for use as a classroom visual aid by teachers of the Mongean method of descriptive geometry to show how the space object corresponds to the representation in the drawing plane. However, it is versatile enough to have uses as a visual aid in other studies of three dimensional space, such as solid geometry, solid analytics, etc.

7. *The kind of mathematics text I would like*, by C. A. Murray, West Texas State Teachers College.

8. Panel Discussion: *What mathematics should be taught during the first four years in college?*

The participants in this discussion were Ina M. Bramblett, Texas Christian University, M. M. Slotnick, Humble Oil and Refining Company, F. E. Ulrich, The Rice Institute, E. C. Klipple, Texas A and M, C. R. Sherer, Texas Christian University.

C. R. SHERER, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The thirty-sixth annual meeting of the Iowa Section of the Mathematical Association of America was held at Drake University, in Des Moines, Iowa, on Friday and Saturday, April 15 and 16, 1949, in conjunction with the Iowa Academy of Science. Professor B. E. Gillam, Vice-Chairman of the Section, presided at the Friday afternoon session; Professor W. M. Davis, Chairman of the Section, presided at the Saturday morning session.

Forty-three persons were present, including the following twenty-eight members of the Association: E. W. Anderson, H. D. Block, Marian E. Daniells, W. M. Davis, R. F. Deniston, R. W. Gardner, R. E. Gaskell, B. E. Gillam, R. N. Goss, Cornelius Gouwens, R. A. Griffin, F. S. Harper, Gertrude A. Herr, J. J. L. Hinrichsen, D. L. Holl, O. C. Kreider, R. J. Lambert, Ta Li, F. M. McGaw,

Martha M. McKelvey, I. F. Neff, A. O. Qualley, Fred Robertson, R. M. Robinson, M. F. Smiley, E. R. Smith, F. M. Stein, H. P. Thielman. A charter member of the Association, Professor I. F. Neff, attended both sessions.

At the business meeting held on Saturday morning the Section elected the following officers for next year: Chairman, Professor B. E. Gillam, Drake University; Vice-Chairman, Professor D. L. Holl, Iowa State College; Secretary, Professor Fred Robertson, Iowa State College.

The following papers were presented at the meeting:

1. *Notes on series with holes in them*, by Professor E. R. Smith, Iowa State College.

It may be shown that

$$H_k(x) = 1 + \frac{x^k}{k!} + \frac{x^{2k}}{2k!} + \cdots$$

$$= \frac{1}{k}\left[e^x + e^{wx} + e^{w^2x} + \cdots + e^{w^{k-1}x}\right],$$

where w is a kth root of unity. The set of functions consisting of $H_k(x)$ and its successive derivatives $H_k'(x)$, $H_k''(x)$, $\cdots$, $H_k^{(k-1)}(x)$ may be considered as generalizations of the hyperbolic functions $\cosh x$ and $\sinh x$. They have a trigonometry and a calculus which corresponds to those which exist for the trigonometric and the hyperbolic functions. They possess addition and double parameter formulas, differential and integral properties, and satisfy a differential equation of the kth order. Most of the elementary results may be obtained by means of the identities

$$[H_k(x) + H_k'(x) + H_k''(x) + \cdots + H_k^{(k-1)}(x)]^2 = H_k(2x) + H_k'(2x) + H_k''(2x) + \cdots + H_k^{(k-1)}(2x),$$

$$H_k(x+y) + H_k'(x+y) + \cdots + H_k^{(k-1)}(x+y)$$
$$= [H_k(x) + H_k'(x) + \cdots + H_k^{(k-1)}(x)][H_k(y) + H_k'(y) + \cdots + H_k^{(k-1)}(y)].$$

2. *Definitions of limits in abstract sets*, by R. F. Deniston, Iowa State College.

The author discussed briefly developments in the generalized theory of limits which led him to his concept of eventual sets, which is to be discussed elsewhere. Contributions of E. H. Moore, H. L. Smith, A. Weil, Garrett Birkhoff, J. W. Tukey, H. Cartan, and Pierre Samuel were outlined.

3. *On primitive functions*, by Ta Li, Drake University.

If the algebraic differential equation of lowest order having $y(x)$ as an integral is of order n, then y is said to be a primitive function of order n. Theorems concerning the order of primitive functions are proved. For instance if y is of order n, and z is an algebraic function of x and y, then z is of order n. If y is of order p, z is of order q, and u is an algebraic function of x, y and z, then u is either of order $p+q$ or of order $p-q$.

4. *Calendar*, by Professor F. M. McGaw, Cornell College.

The author discussed relations between the rotation and orbital motion of the earth, and efforts made to reconcile these, and obtain therefrom a calendar which would serve both civil and religious purposes. The Julian reformation and the Gregorian adjustment, by dropping days and changing date of year's beginning, were reviewed. The device of "leap year" was described.

The motion of the moon and the device of extra months employed by certain ancient nations, and by some of the Oriental groups in recent years, to better relate this motion to that of the sun were discussed. The meaning of Metonic cycle, golden number and epact was stated. The relation of these quantities to the calculation of religious festivals, particularly Easter, were discussed and illustrated. Formulae for these calculations were illustrated.

The principles involved in the modern proposal of a world calendar, particularly the so-called equal-quarter twelve months calendar of the Calendar Reform Association, were discussed.

5. *Report of the Secretary*, by Fred Robertson, Iowa State College.

The secretary reported on the meetings of the section secretaries at the Madison and Columbus meetings of the Association, and gave some statistics on the sections.

6. *Applications of the Laplace transformations*, by Professor R. E. Gaskell, Iowa State College.

After introductory remarks concerning the Laplace transformation, with applications to systems of ordinary differential equations, an application to a simple boundary value problem was used to illustrate the procedure for solution by transformation. Then applications were made to the problems of temperature in a bar in contact with a finite quantity of stirred liquid, of longitudinal vibration of a bar with attached mass, and of the vibration of a cantilever with a dashpot at the free end. These problems ordinarily require expansions in terms of a set of functions which are not orthogonal, because of the presence of the parameter in the boundary conditions of the characteristic value problem. Finally, an application of the Laplace transformation to a control problem was made. This problem was such that separation of variables would lead to a characteristic value system which is not self-adjoint. The transformation method gave the solution directly.

7. *On generalized means*, by Professor H. P. Thielman, Iowa State College.

The mean $M_f[x_i; c_i]$ of the numbers x_i with respect to a strictly monotone function $f(x)$ and with weights c_i was defined as

$$f^{(-1)}\left[\sum_{i=1}^{n} c_i f(x_i)\right],$$

where $f^{(-1)}[x]$ stands for the inverse function of $f(x)$, and where the c_i are such that $\sum_{i=1}^{n} c_i = 1$. It was shown that if a function $F(x)$ is such that

$$F\{M_f[x_i; c_i]\} = M_g[F(x_i), k_i],$$

then $F(x) = g^{(-1)}[Af(x)+B]$, and $c_i = k_i$, where A and B are constants. A number of examples were given to illustrate this general result. (J. Aczel, *Commentarii Mathematici Helvitici*, vol. 21, 1948, pp. 247–252.) Applications were made to convex solutions of functional equations.

8. *Short proof of the gramian inequality*, by Professor Bernard Vinograde, Iowa State College, introduced by the Secretary.

A short proof is given of the gramian inequality (see Turnbull and Aitken, *Canonical Matrices*, p. 98) based on the triangular factorization of the gramian matrix.

9. *Escalation with multiple roots*, by R. J. Lambert, Iowa State College.

The purpose of this paper is to complete the Morris and Head escalator process for finding the characteristic roots and vectors of a square matrix. The modified process consists of finding a transformation matrix P_0 by using all known vectors and solving for the unknown columns which correspond to the multiple roots. Thus the principal sub-matrix A_p of the matrix A_n is transformed to a matrix A_0 which is diagonal except for a few ones or zeros in the super-diagonal. Therefore the left hand side of the equation

$$|(A_{p+1} - I_{p+1})| = \left| \begin{pmatrix} P_0^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_p & D \\ c & a_{p+1,p+1} \end{pmatrix} \begin{pmatrix} P_0 & 0 \\ 0 & 1 \end{pmatrix} - \lambda I_{p+1} \right|$$

$$= \left| \begin{pmatrix} A_0 & -\lambda I_p & P_0^{-1}D \\ c & P_0 & a_{p+1,p+1}-\lambda \end{pmatrix} \right| = 0$$

will be a determinant which can be expanded in terms of the characteristic roots of a previous stage, and which can easily be put into the partial fraction form by dividing by $|(A_0-\lambda I_p)|$. This method reverts to the Morris method if all characteristic roots are distinct at every stage.

10. *A review of certain problems facing the teacher of mathematics and science*, by Professor C. F. Reid, Dubuque, Iowa, introduced by the Secretary.

The speaker discussed the trends in mathematics teaching in Iowa during the years immediately preceding the outbreak of the second world war, during the war, and at present. He emphasized the way in which the new course of study for the state attempted to meet the needs of the 80 per cent of high school pupils who do not plan to attend college, as well as the 20 per cent who do.

11. *The number of optically inactive glycols*, by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

Glycols being organic compounds with two *OH* radicals in each molecule, the number of optically inactive ones with a given number of carbon atoms is sought. These are the molecules with a plane of symmetry—a plane which may be the perpendicular bisector of the segment joining the two oxygen atoms or which may contain them. The enumeration is made by methods developed by Harvey Diehl and the author, and by the use of data obtained by Russell E. Carr for optically inactive alcohols.

12. *Properties of an integral transformation*, by H. D. Block, Iowa State College.

The author considers the family of integral transformations, of a suitable function $g(y)$, defined by

$$(1) \qquad f_n(k, x) = \int_{-\infty}^{\infty} |x-y|^n e^{-k|x-y|} g(y)dy = L_n\{g(y)\}, \qquad k > 0, n = 0, 1, 2, \cdots.$$

The inversion formula

$$\lim_{k\to\infty} k^{n+1} L_n\{g(y)\} = n![g(x+) + g(x-)]$$

is derived.

Since $f_n(k, x+a) = L_n\{g(y+a)\}$, it follows that $L_n\{g'(y)\} = D_x L_n\{g(y)\}$, where $g(y) \in C$ and $\in C_1$ in sections. Corresponding formulas are given when $g(y)$ and its derivatives have finite jumps. The formula for $L_m\{L_n\} = L_n\{L_m\}$ is given as a simple linear combination of $L_0, L_1, \cdots, L_{m+n+1}$. Thus L_i can be written as a linear combination of the iterates of L_0, showing that all the eigenfunctions of L_0 (viewing (1) as an integral equation in which g may involve the parameter k) are also eigenfunctions of L_i, but associated with different eigenvalues. Other properties of the transformation are also studied, and illustrations given of its application to special functional equations.

13. *Rational canonical form of a matrix*, by Professor M. F. Smiley, State University of Iowa, by title.

This paper appears in this issue of the MONTHLY.

14. *Actual construction of fundamental systems of general linear differential equations and the behaviour of such systems*, by Ta Li, Drake University.

Let $f_1, f_2, f_3, \cdots, f_n$ and f be given functions in (a, b). Writing D for d/dx and $K(x, D)$ for $\sum_0 \lambda^{n-\lambda} f_{n-\lambda}(x) D^\lambda$, the general linear differential equation takes the form

$$D''y - K(x, D)y = f(x). \tag{1}$$

For any given function $g(x)$ we define:

$$F_1(x, g) = \int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-1}} K(t_n, D)g(t_n)dt_n \cdots dt_1,$$
$$F_{n+1}(x, g) = \int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-1}} K(t_n, D)F_n(t_n, g)dt_n \cdots dt_1, \tag{2}$$

where $n = 1, 2, 3, \cdots$, provided the integrals exist. The functions

$$y_{\mu+1}(x) = \frac{(x-a)^\mu}{\mu!} + \sum_1^\infty kF_k\left(x, \frac{(t-c)^\mu}{\mu}\right) \quad \mu = 0, 1, 2, 3, \cdots, n-1 \tag{3}$$

are n linearly independent solutions of the homogeneous equation if the series are uniformly convergent.

The general solution of (1) is given symbolically by

$$Y = \sum_0^{n-1} \mu C_{\mu+1} Y_{\mu+1} + D^{-n} \frac{1}{1 - K(x, D)D^{-n}} f(x).$$

Different conditions are imposed on $f_1, f_2, \cdots, f_n$ and f to study the behavior of the solutions.

15. *Bernoulli numbers. Their origin and development*, by Professor R. B. McClenon, Grinnell College, by title.

FRED ROBERTSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Annual Meeting, New York City, December 30, 1949.

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, April 29, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, Spring, 1950.

KENTUCKY

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, Spring, 1950.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Fall, 1949.

METROPOLITAN NEW YORK, Spring, 1950.

MICHIGAN, March, 1950.

MINNESOTA, University of North Dakota, Grand Forks, October 15, 1949.

MISSOURI, Spring, 1950.

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA, Berkeley, January 28, 1950.

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA, Oklahoma City, October 14, 1949.

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950.

PHILADELPHIA, Haverford College, November 26, 1949.

ROCKY MOUNTAIN, University of Denver, April, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950.

SOUTHWESTERN, Spring, 1950.

TEXAS, Abilene, Spring, 1950.

UPPER NEW YORK STATE, Syracuse University, Spring, 1950.

WISCONSIN, Marquette University, Milwaukee, May, 1950.

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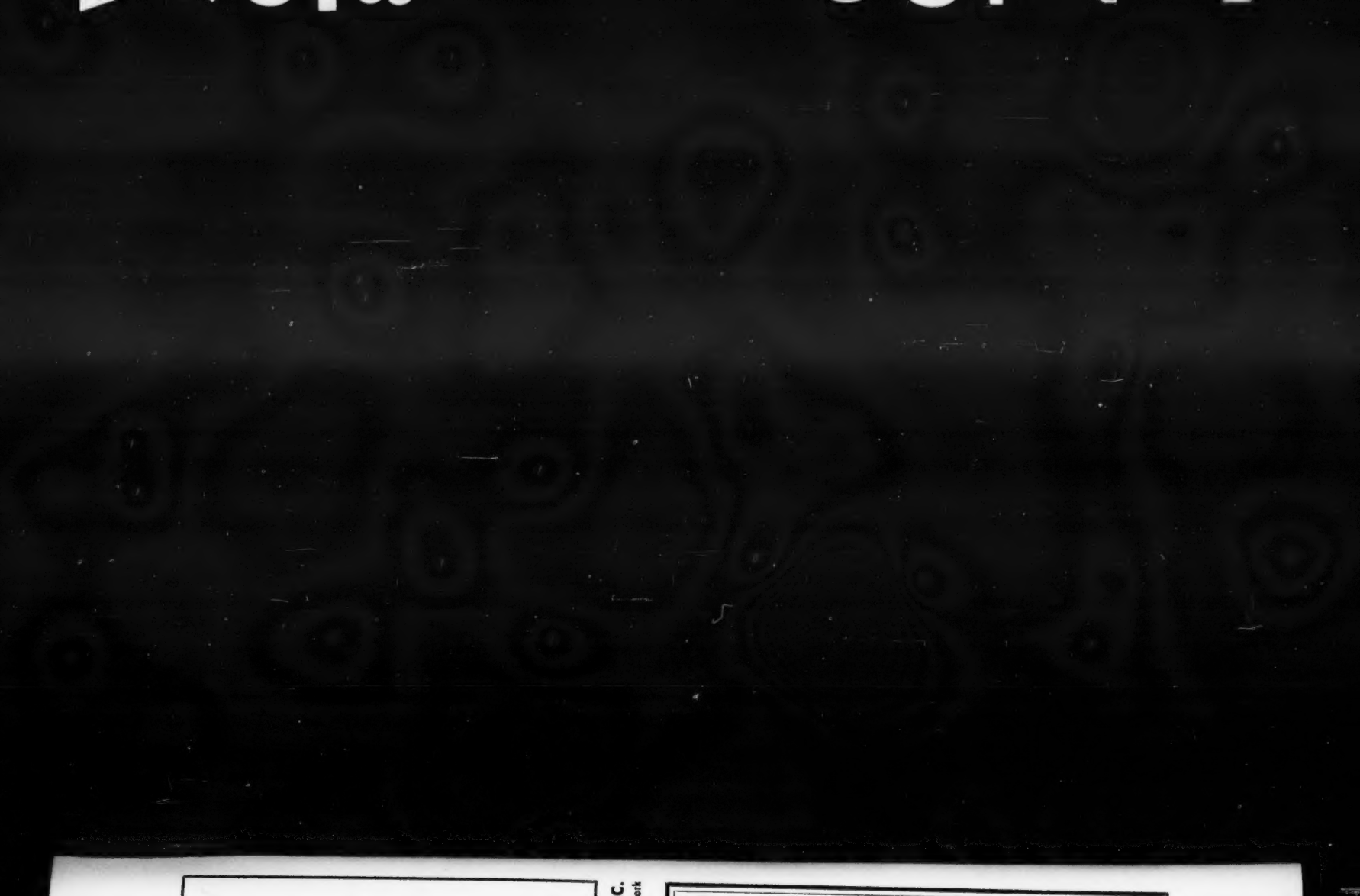

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